¡!DOCTYPE HTML PUBLIC”-//W3C//DTD HTML 4.0 Final//EN” ¿ ¡HTML¿ ¡HEAD¿ ¡LINK REL=”STYLESHEET” TYPE=”text/css” HREF=”/global$_c$ss/style.css”TITLE = ”Style” >< METANAME = ”ROBOTS”CONTENT = ”ALL” >< METANAME = ”description”CONTENT = ”fancyheadings.sty.ComputationalChemistryList.ResourceforComputationalChemist METANAME = ”keywords”CONTENT = ”CCL, OSC, NSF, abinitio, archives, chemicalbonding, chemicalphysics TITLE > fancyheadings.sty < /TITLE >< /HEAD >< BODYBGCOLOR = ”FFFFFF”CLASS = ”white” >< AHREF = ”/chemistry/index.shtml” >< IMGSRC = ”/global$_i$mages/triccl.gif”ALIGN = BOTTOMWIDTH = ”326”BORDER = 0HEIGHT = ”80”ALT = ”CCLHomePage” >< /A >< TABLEWIDTH = ”100 < TR >< TDCLASS = ”botlink”WIDTH = ”100http : //www.ccl.net/cca/software/SOURCES/FORTRAN/mopac7$_s$ources/mopac−uncompressed−manuals/mopac−man−latex−source/fancyheadings.sty.shtml < /TD >< /TR >< /TABLE >< TABLEWIDTH = ”100 < TR >< TDCLASS = ”up”WIDTH = ”143” >< AHREF = ”../index.shtml”CLASS = ”navbig” >< IMGSRC = ”/global$_i$mages/upbut.gif”ALT = ”UpDirectory”BORDER = ”0” >< /A >< /TD >< TDCLASS = ”title” >< EMCLASS = ”title” > CCL < /EM > fancyheadings.sty < /TD >< /TR >< TR >< TDCLASS = ”nav”VALIGN = ”top”NOWRAP >< TABLEWIDTH = ”143”CELLSPACING = ”0”CELLPADDING = ”0” >< TR >< TDCLASS = ”subtab”WIDTH = ”143” >< TABLEWIDTH = ”140”BORDER = ”3”CELLSPACING = ”0”CELLPADDING = ”2” >< TR >< TDCLASS = ”head”WIDTH = ”143” >< AHREF = ”index.shtml”CLASS = ”navbig” > mopac − man − latex − source < /A >< /TD >< /TR >< TR >< TDCLASS = ”list” >< AHREF = ”fancyheadings.st.shtml”CLASS = ”navsmall” > fancyheadings.st < /A >, < AHREF = ”fancyheadings.sty.shtml”CLASS = ”navsmall” > fancyheadings.sty < /A >, < AHREF = ”front.shtml”CLASS = ”navsmall” > front < /A >, < AHREF = ”front.tex.shtml”CLASS = ”navsmall” > front.tex < /A >, < AHREF = ”manual.shtml”CLASS = ”navsmall” > manual.csh < /A >, < AHREF = ”mopac.aux”CLASS = ”navsmall” > mopac.aux < /A >, < AHREF = ”mopac.bbl.shtml”CLASS = ”navsmall” > mopac.bbl < /A >, < AHREF = ”mopac.blg.shtml”CLASS = ”navsmall” > mopac.blg < /A >, < AHREF = ”mopac.dvi”CLASS = ”navsmall” > mopac.dvi < /A >, < AHREF = ”mopac.idx.shtml”CLASS = ”navsmall” > mopac.idx < /A >, < AHREF = ”mopac.ilg.shtml”CLASS = ”navsmall” > mopac.ilg < /A >, < AHREF = ”mopac.ind.shtml”CLASS = ”navsmall” > mopac.ind < /A >, < AHREF = ”mopac.tex”CLASS = ”navsmall” > mopac.tex < /A >, < AHREF = ”polar.shtml”CLASS = ”navsmall” > polar.tex < /A >, < AHREF = ”referenc.shtml”CLASS = ”navsmall” > referenc < /A >, < /TD >< /TR >< /TABLE >< /TD >< /TR >< /TABLE >< P >< /P >

¡/TD¿ ¡TD CLASS=”body” VALIGN=”top”¿ ¡PRE¿

¡/PRE¿ ¡/TD¿ ¡/TR¿ ¡/TABLE¿ ¡P CLASS=”links”¿ [¡A HREF=”/chemistry/index.shtml” CLASS=”navsmall”¿CCL Home Page¡/A¿] ¡BR¿ [¡A HREF=”index.shtml” CLASS=”navsmall”¿mopac-man-latex-source¡/A¿] ¡BR¿ [¡A HREF=”fancyheadings.sty” CLASS=”navsmall”¿Raw Version of this page¡/A¿] ¡/P¿ ¡TABLE WIDTH=”100¡TR¿ ¡TD CLASS=”modtime” align=”middle”¿ Modified:

Wed Jul 5 16:00:00 1995 GMT ¡/TD¿ ¡/TR¿ ¡TR¿ ¡TD CLASS=”counter” align=”middle”¿ Page accessed 286 times since Sat Apr 17 22:31:03 1999 GMT ¡/TD¿ ¡/TR¿ ¡/TABLE¿ ¡/BODY¿ ¡/HTML¿

Global existence of solutions to the Einstein-Yang-Mills-dilaton equations *

Todd A. Oliynyk †
School of Mathematics and Statistics
University of Canberra
ACT 2601, Australia

Abstract

We prove the existence of a countable number of solutions to the static spherically symmetric $SU(2)$ Einstein-Yang-Mills-dilaton (EYMd) equations. Existence is established using a Newtonian limit type argument which shows that static spherically symmetric $SU(2)$ Yang-Mills-dilaton solutions can be continued smoothly to EYMd solutions provided they satisfy certain fall off conditions.

1 Introduction

Unlike the four dimensional Yang-Mills (YM) equations which have no static solutions of finite energy [7, 6], the $SU(2)$ Yang-Mills-dilaton (YMd) equations were shown numerically to posses a countably infinite sequence of static, globally regular, spherically symmetric solutions [12, 3]. Existence of these solutions was rigorously established using shooting techniques in [10]. In the papers [13, 4] it was found, again numerically, that the YMd solutions persist when the Yang-Mills and dilaton fields are coupled to gravity. The result is a countably infinite sequence of static, globally regular, spherically symmetric solutions to the $SU(2)$ Einstein-Yang-Mills-dilaton (EYMd) equations with the same qualitative behavior for the Yang-Mills and dilaton fields as when gravity is absent.

In this paper we rigorously prove the existence of a countable number of solutions to the static spherically symmetric $SU(2)$ EYMd equations. We prove existence by using a Newtonian limit type argument to show that the YMd solutions can be continued smoothly to EYMd solutions provided they satisfy certain fall off conditions.

The Newtonian limit argument in the form that is employed in this paper was developed by Lottermoser in [14] and subsequently used by Heilig to establish the existence of slowly rotating stars [11]. The results of Heilig and of this paper show that Newtonian limit is a powerful method for obtaining existence theorems in general relativity for static or stationary matter models. We will, in upcoming work, present a detailed description of the Newtonian limit method along with some applications so that the technique may find wider applications.

*2000 *Mathematics Subject Classification* Primary 83C20, 83C22; Secondary 53C30.

†todd.oliynyk@ise.canberra.edu.au

In section 2 we set up the equations in a form suitable to use the Newtonian limit while in section 3 we review the theory of weighted Sobolev spaces which will be essential to our existence proof. The Banach spaces for our field variables (i.e. the dilaton field, gauge potential, and metric density) are set up in section 4 and then in section 5 the field equations are shown to be smooth on those spaces. Sections 6-8 contain the Newtonian limit argument. In these sections it is shown that if there exist a static spherically symmetric solution to the YMd equations satisfying certain conditions then the solution can be continued smoothly to a solution of the full EYMd equations. Finally, global existence is established in section 9 by showing that the solutions to the YMd equations that were established rigorously in [10] satisfy our conditions and hence produce EYMd solutions.

2 EYMd equations

For indexing of tensors and related quantities greek indices, α, β, γ etc., will always run from 0 to 4 while roman indices, i, j, k etc., will range from 1 to 3. We will use bold letters such as $\mathbf{x}$ to denote points in $\mathbb{R}^3$, i.e. $\mathbf{x} = (x^1, x^2, x^3)$.

Let $\underset{o}{g}$ denote the Minkowski metric on $\mathbb{R}^4$. Fix a global coordinate system (x^0, x^1, x^2, x^3) so that

$$\underset{o}{g}_{\alpha\beta} = \mathrm{diag}(-\lambda^{-1}, 1, 1, 1) \quad \lambda := \frac{1}{c^2} \tag{2.1}$$

where c is the speed of light. Define $\underset{o}{g}^{\alpha\beta}$ by $(\underset{o}{g}^{\alpha\beta}) := (\underset{o}{g}_{\alpha\beta})^{-1}$ which gives

$$\underset{o}{g}^{\alpha\beta} = \mathrm{diag}(-\lambda, 1, 1, 1)\,. \tag{2.2}$$

Define the Minkowski metric density

$$\underset{o}{\mathfrak{g}}^{\alpha\beta} := |\underset{o}{g}|^{\frac{1}{2}} \underset{o}{g}^{\alpha\beta} \quad \text{where} \quad |\underset{o}{g}| := |\det(\underset{o}{g}_{\alpha\beta})|\,. \tag{2.3}$$

Assume that $g_{\alpha\beta}$ is another metric defined on $\mathbb{R}^4$. Let $(g^{\alpha\beta}) := (g_{\alpha\beta})^{-1}$ and introduce the density

$$\mathfrak{g}^{\alpha\beta} := |g|^{\frac{1}{2}} g^{\alpha\beta} \quad \text{where} \quad |g| := |\det(g_{\alpha\beta})|\,. \tag{2.4}$$

Following Lottermoser [14], we form the tensor density

$$\mathfrak{U}^{\alpha\beta} := \frac{1}{4\lambda^{\frac{3}{2}}}(\mathfrak{g}^{\alpha\beta} - \underset{o}{\mathfrak{g}}^{\alpha\beta}) \tag{2.5}$$

which will be taken as our primary gravitational variable. Observe that the metric $g^{\alpha\beta}$ can be recovered from $\mathfrak{U}^{\alpha\beta}$ by

$$g^{\alpha\beta} = \frac{1}{\sqrt{|g|}}\mathfrak{g}^{\alpha\beta}$$

where $\mathfrak{g}^{\alpha\beta} = \underset{o}{\mathfrak{g}}^{\alpha\beta} + 4\lambda^{\frac{3}{2}}\mathfrak{U}^{\alpha\beta}$ and $|g| = |\det(\mathfrak{g}^{\alpha\beta})|$.

The Einstein equations can be written in terms of the density (2.5) as [14],

$$4\pi G|\mathfrak{d}|T^{\alpha\beta} = A^{\alpha\beta} + B^{\alpha\beta} + C^{\alpha\beta} + D^{\alpha\beta}\,, \tag{2.6}$$

where

$$\bar{\underset{o}{\mathfrak{g}}}^{\alpha\beta} := \sqrt{\lambda}\underset{o}{\mathfrak{g}}^{\alpha\beta}, \tag{2.7}$$

$$\bar{\underset{o}{\mathfrak{g}}}_{\alpha\beta} := \sqrt{\lambda}\underset{o}{\mathfrak{g}}_{\alpha\beta} \quad \text{where} \quad (\underset{o}{\mathfrak{g}}_{\alpha\beta}) := (\underset{o}{\mathfrak{g}}^{\alpha\beta})^{-1}, \tag{2.8}$$

$$\bar{\mathfrak{g}}^{\alpha\beta} := \sqrt{\lambda}\mathfrak{g}^{\alpha\beta} = \bar{\underset{o}{\mathfrak{g}}}^{\alpha\beta} + 4\lambda^2\mathfrak{U}^{\alpha\beta}, \tag{2.9}$$

$$\bar{\mathfrak{g}}_{\alpha\beta} := \sqrt{\lambda}\mathfrak{g}_{\alpha\beta} \quad \text{where} \quad (\mathfrak{g}_{\alpha\beta}) := (\mathfrak{g}^{\alpha\beta})^{-1}\,, \tag{2.10}$$

$$\mathfrak{d} := \lambda\det(\mathfrak{g}^{\alpha\beta}), \tag{2.11}$$

$$A^{\alpha\beta} := 2\left(\tfrac{1}{2}\bar{\mathfrak{g}}_{\mu\nu}\bar{\mathfrak{g}}_{\gamma\rho} - \bar{\mathfrak{g}}_{\rho\mu}\bar{\mathfrak{g}}_{\gamma\nu}\right)\left(\bar{\mathfrak{g}}^{\alpha\kappa}\bar{\mathfrak{g}}^{\beta\sigma} - \tfrac{1}{2}\mathfrak{g}^{\alpha\beta}\bar{\mathfrak{g}}^{\kappa\sigma}\right)\mathfrak{U}^{\mu\nu}{}_{,\kappa}\mathfrak{U}^{\gamma\rho}{}_{,\sigma}, \tag{2.12}$$

$$B^{\alpha\beta} := 4\lambda\bar{\mathfrak{g}}_{\kappa\sigma}\left(2\bar{\mathfrak{g}}^{\gamma(\alpha}\mathfrak{U}^{\beta)\sigma}{}_{,\rho}\mathfrak{U}^{\kappa\rho}{}_{,\gamma} - \tfrac{1}{2}\bar{\mathfrak{g}}^{\alpha\beta}\mathfrak{U}^{\kappa}{}_{\rho\gamma}\mathfrak{U}^{\sigma\gamma}{}_{\rho} - \bar{\mathfrak{g}}^{\gamma\rho}\mathfrak{U}^{\alpha\kappa}{}_{,\gamma}\mathfrak{U}^{\beta\sigma}{}_{\rho}\right), \tag{2.13}$$

$$C^{\alpha\beta} := 4\lambda^2\left(\mathfrak{U}^{\alpha\beta}{}_{,\kappa}\mathfrak{U}^{\kappa\rho}{}_{,\rho} - \mathfrak{U}^{\alpha\kappa}{}_{,\rho}\mathfrak{U}^{\beta\rho}{}_{,\kappa}\right), \tag{2.14}$$

$$D^{\alpha\beta} := \bar{\mathfrak{g}}^{\mu\nu}\mathfrak{U}^{\alpha\beta}{}_{,\mu\nu} + \bar{\mathfrak{g}}^{\alpha\beta}\mathfrak{U}^{\mu\nu}{}_{,\mu\nu} - 2\mathfrak{U}^{\mu(\alpha}{}_{,\mu\nu}\bar{\mathfrak{g}}^{\beta)\nu}, \tag{2.15}$$

and $T^{\alpha\beta}$ is the stress-energy tensor. As discussed in [11], any solution $(\lambda, \mathfrak{U}^{\alpha\beta}, T^{\alpha\beta})$ of (2.6) for $\lambda > 0$ is a solution of Einsteins equations displayed in units where $c = 1/\sqrt{\lambda}$. Following [11], we choose harmonic coordinates

$$\nabla_\alpha\nabla^\alpha x^\beta = 0\,, \quad \text{or equivalently} \quad \mathfrak{U}^{\alpha\beta}{}_{,\beta} = 0\,,$$

which allows us to write the full Einstein field equations as

$$\mathfrak{U}^{\alpha\beta}{}_{,\beta} = 0\,, \tag{2.16}$$

$$4\pi G|\mathfrak{d}|T^{\alpha\beta} = E^{\alpha\beta}\,, \tag{2.17}$$

where

$$\begin{aligned} E^{\alpha\beta} := \underset{o}{\bar{\mathfrak{g}}}^{\mu\nu}\mathfrak{U}^{\alpha\beta}{}_{,\mu\nu} + 4\lambda^2\left(\mathfrak{U}^{\mu\nu}\mathfrak{U}^{\alpha\beta}{}_{,\mu\nu} + \mathfrak{U}^{\alpha\beta}\mathfrak{U}^{\mu\nu}{}_{,\mu\nu} - 2\mathfrak{U}^{\mu(\alpha}{}_{,\mu\nu}\mathfrak{U}^{\beta)\nu}\right) \\ + A^{\alpha\beta} + B^{\alpha\beta} + C^{\alpha\beta}\,. \end{aligned} \tag{2.18}$$

The equations (2.17) will be called the *reduced field equations*.

It is important to recognize that alone reduced field equations (2.17) are not equivalent to the Einstein field equations (2.6). However, it is shown in [11] §6 that if $T^{\alpha\beta}{}_{;\beta} = 0$ and (2.17) can be solved and the stress-energy tensor $T^{\alpha\beta}$ satisifes certain conditions then the harmonic condition (2.16) will be automatically satisfied. In this case, a solution to (2.17) will actually be a solution to the full Einstein equation (2.6).

We will let $A = A_\alpha dx^\alpha$ denote the $SU(2)$-gauge potential and ψ the dilaton field. The SU(2) Yang-Mills-dilation equations are

$$D^\alpha\left(e^{2\kappa\psi}F_{\alpha\beta}\right) = 0\,, \tag{2.19}$$

$$\nabla^\alpha\nabla_\alpha\psi - \frac{\kappa\ell_Y}{\ell_d}e^{2\kappa\psi}g^{\alpha\beta}g^{\mu\nu}\langle F_{\alpha\mu}|F_{\beta\nu}\rangle = 0\,, \tag{2.20}$$

where $D_\alpha(\cdot) := \nabla_\alpha(\cdot) + [A_\alpha, \cdot]$ is the gauge covariant derivative, ℓ_Y the Yang-Mills coupling constant, $\{\ell_d, \kappa\}$ the dilaton coupling constants,

$$F_{\alpha\beta} := A_{\beta,\alpha} - A_{\alpha,\beta} + [A_\alpha, A_\beta] \tag{2.21}$$

the gauge field, and $\langle\cdot|\cdot\rangle$ is an Ad-invariant positive definite inner-product on $\mathfrak{su}(2)$. Multiplying (2.19) and (2.20) by $\sqrt{\lambda|g|}$ and $\lambda|g|$, respectively, we find that

$$\bar{\mathfrak{g}}^{\alpha\nu}\left(F_{\alpha\beta,\nu} - \Gamma^\mu_{\alpha\nu}F_{\mu\beta} - \Gamma^\mu_{\beta\nu}F_{\alpha\mu} + 2\kappa\psi_{,\nu}F_{\alpha\beta} + [A_\nu, F_{\alpha\beta}]\right) = 0\,, \tag{2.22}$$

$$\bar{\mathfrak{g}}^{\alpha\beta}\left(\psi_{,\alpha\beta} - \Gamma^\mu_{\alpha\beta}\psi_{,\mu} - \frac{\kappa\ell_Y}{\ell_d}\frac{e^{2\kappa\psi}}{\sqrt{|\mathfrak{d}|}}\bar{\mathfrak{g}}^{\mu\nu}\langle F_{\alpha\mu}|F_{\beta\nu}\rangle\right) = 0\,, \tag{2.23}$$

where the Christoffel $\Gamma^\alpha_{\beta\gamma}$ symbols are given by

$$\Gamma^\alpha_{\beta\gamma} = \bar{\mathfrak{g}}^{\alpha\mu}(2\bar{\mathfrak{g}}_{\beta\sigma}\bar{\mathfrak{g}}_{\gamma\tau} - \bar{\mathfrak{g}}_{\beta\gamma}\bar{\mathfrak{g}}_{\sigma\tau})\mathfrak{U}^{\sigma\tau}{}_{,\mu} + 2\lambda(\bar{\mathfrak{g}}_{\sigma\tau}\delta^\alpha_{(\beta}\mathfrak{U}^{\sigma\tau}{}_{,\gamma)} - 2\bar{\mathfrak{g}}_{\sigma(\beta}\mathfrak{U}^{\alpha\sigma}{}_{,\gamma)})\,. \tag{2.24}$$

The stress energy tensor can be written as

$$\begin{aligned} T^{\alpha\beta} = {} & \tfrac{1}{2}\ell_d\left(g^{\alpha\mu}g^{\beta\nu}\psi_{,\mu}\psi_{,\nu} - \tfrac{1}{2}g^{\alpha\beta}g^{\mu\nu}\psi_{,\mu}\psi_{,\nu}\right) + \\ & \ell_Y e^{2\kappa\psi}\left(g^{\alpha\mu}g^{\beta\nu}g^{\sigma\tau}\langle F_{\mu\sigma}|F_{\nu\tau}\rangle - \tfrac{1}{4}g^{\mu\nu}g^{\sigma\tau}g^{\alpha\beta}\langle F_{\mu\sigma}|F_{\nu\tau}\rangle\right)\,. \end{aligned} \tag{2.25}$$

Using the YMd equations (2.19)-(2.20) , it is straightforward to verify that any YMd solution satisfies

$$T^{\alpha\beta}{}_{;\beta} = 0 \tag{2.26}$$

automatically irrespective of the metric. Consequently, it will be enough to solve the reduced field equations (2.17) and the YMd equations (2.19)-(2.20) to obtain a solution to the full EYMd field equations.

Let

$$\mathcal{T}^{\alpha\beta} := 4\pi G|\mathfrak{d}|T^{\alpha\beta} \tag{2.27}$$

so that

$$\begin{aligned} \mathcal{T}^{\alpha\beta} = {} & 2\pi G\ell_d\left(\bar{\mathfrak{g}}^{\alpha\mu}\bar{\mathfrak{g}}^{\beta\nu}\psi_{,\mu}\psi_{,\nu} - \tfrac{1}{2}\bar{\mathfrak{g}}^{\alpha\beta}\bar{\mathfrak{g}}^{\mu\nu}\psi_{,\mu}\psi_{,\nu}\right) + \\ & 4\pi G\frac{\ell_Y}{\sqrt{|\mathfrak{d}|}}e^{2\kappa\psi}\left(\bar{\mathfrak{g}}^{\alpha\mu}\bar{\mathfrak{g}}^{\beta\nu}\bar{\mathfrak{g}}^{\sigma\tau}\langle F_{\mu\sigma}|F_{\nu\tau}\rangle - \tfrac{1}{4}\bar{\mathfrak{g}}^{\mu\nu}\bar{\mathfrak{g}}^{\sigma\tau}\bar{\mathfrak{g}}^{\alpha\beta}\langle F_{\mu\sigma}|F_{\nu\tau}\rangle\right)\,. \end{aligned} \tag{2.28}$$

3 Weighted Sobolev Spaces

Let V denote a finite dimensional vector space with norm $|\cdot|$.

Definition 3.1. *The* weighted Lesbegue space $L^p_\delta(\mathbb{R}^n, V)$, $1 \le p \le \infty$, *with weight* $\delta \in \mathbb{R}$ *is the set of all measurable maps from* $\mathbb{R}^n$ *to* V *in* $L^p_{loc}(\mathbb{R}^n, V)$ *such that the norm*

$$\|u\|_{p,\delta} = \begin{cases} \left(\int_{\mathbb{R}^n} |u|^p\sigma^{-\delta p - n}d^nx\right)^{\frac{1}{p}} & \text{if } p < \infty \\ \operatorname{ess\,sup}_{\mathbb{R}^n}(\sigma^{-\delta}|u|) & \text{if } p = \infty\,, \end{cases}$$

is finite. Here $\sigma(\mathbf{x}) := \sqrt{|x|^2 + 1}$. *If* $V = \mathbb{R}$ *then we write* $L^p_\delta(\mathbb{R}^n)$ *instead of* $L^p_\delta(\mathbb{R}^n, V)$.

Definition 3.2. *The* weighted Sobolev space $W^{k,p}_\delta(\mathbb{R}^n, V)$*,* $1 \le p \le \infty$*,* $k \in \mathbb{N}_0$*, with weight* $\delta \in \mathbb{R}$ *is the set*

$$W^{k,p}_\delta(\mathbb{R}^n, V) := \{\, u \in L^p_\delta(\mathbb{R}^n, V) \,|\, \partial^I u \in L^p_{\delta-|I|}(\mathbb{R}^n, V) \text{ for all } I : |I| \le k \,\}$$

with norm

$$\|u\|_{k,p,\delta} := \sum_{|I|\le k} \|\partial^I u\|_{p,\delta-|I|} \,,$$

where $I = (I_1, I_2, \ldots, I_n)$ *is a multi-index and* $\partial^I := \partial_1^{I_1}\partial_2^{I_2}\cdots\partial_n^{I_n}$*. If* $V = \mathbb{R}$ *then we will write* $W^{k,p}_\delta(\mathbb{R}^n)$ *instead of* $W^{k,p}_\delta(\mathbb{R}^n, V)$*.*

From the definition, it is clear that differentiation

$$\partial_j : \mathrm{W}^{k,p}_\delta(\mathbb{R}^n, V) \to \mathrm{W}^{k-1,p}_{\delta-1}(\mathbb{R}^n, V) \tag{3.1}$$

is a continuous linear map. Also from the definition and Hölders inequality it is easy to show (see also [1], proposition 1.2 (i)) that if $k_1 \ge k_2$ and $\delta_1 < \delta_2$ then

$$\mathrm{W}^{k_1,p}_{\delta_1}(\mathbb{R}^n, V) \subset \mathrm{W}^{k_2,p}_{\delta_2}(\mathbb{R}^n, V)\,. \tag{3.2}$$

Finally, we note that the set $\mathrm{C}^\infty_0(\mathbb{R}^n, V)$ of smooth maps from $\mathbb{R}^n$ to V with compact support is dense in $\mathrm{W}^{k,p}_\delta(\mathbb{R}^n, V)$. As above, if $V = \mathbb{R}$ then we write $\mathrm{C}^\infty_0(\mathbb{R}^n)$ instead of $\mathrm{C}^\infty_0(\mathbb{R}^n, V)$. We will now state some results in weighted Sobolev spaces that will be needed. For proofs see [1] and [5].

Lemma 3.3. *If there exists a multiplication* $V_1 \times V_2 \to V_3$ $(u, v) \mapsto u \cdot v$ *then the corresponding multiplication*

$$W^{k_1,p}_{\delta_1}(\mathbb{R}^n, V_1) \times W^{k_2,p}_{\delta_2}(\mathbb{R}^n, V_2) \to W^{k_3,p}_{\delta_3}(\mathbb{R}^n, V_3) \;:\; (u, v) \mapsto u \cdot v$$

is bilinear and continuous if $k_1, k_2 \ge k_3$*,* $k_3 < k_1 + k_2 - n/p$*, and* $\delta_1 + \delta_2 < \delta_3$ *.*

Proof. See lemma 2.5 in [5] for the case $p = 2$. For all p this can be proved easily using theorem 1.2 of [1]. □

Theorem 3.4. *For* $\delta < 0$ *the Laplacian*

$$\Delta : W^{k,p}_\delta(\mathbb{R}^n, V) \to W^{k-2,p}_{\delta-2}(\mathbb{R}^n, V)$$

is continuous and injective. Moreover if $2 - n < \delta < 0$ *then the Laplacian is an isomorphism. The inverse is given by*

$$(\Delta^{-1}u)(\mathbf{x}) = \frac{1}{(2-n)\omega_n} \int_{\mathbb{R}^n} \frac{u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{(n-2)}} d^n y \,, \tag{3.3}$$

where ω_n *is the area of the unit sphere in* $\mathbb{R}^n$*.*

Lemma 3.5. *For* $k_1 > k_2$*,* $\delta_1 < \delta_2$*, and* $1 \le p < \infty$ *the embedding* $W^{k_1,p}_{\delta_1}(\mathbb{R}^n, V) \to W^{k_2,p}_{\delta_2}(\mathbb{R}^n, V)$ *is compact.*

4 Static spherically symmetric fields

We assume that all the fields are static and that ∂_0 is a timelike hypersurface orthogonal killing vector field for the metric. Therefore

$$\partial_0 \mathfrak{U}^{\alpha\beta} = 0\,,\ \partial_0 A_\alpha = 0\,,\ \partial_0 \psi = 0 \quad \text{and} \quad \mathfrak{U}^{j0} = \mathfrak{U}^{0j} = 0\,.$$

Since $\mathfrak{U}^{\alpha\beta}$ is symmetric, i.e. $\mathfrak{U}^{\alpha\beta} = \mathfrak{U}^{\beta\alpha}$, we define the following subspace of the 4 by 4 matrices

$$\mathbb{S} := \{\, X = (X^{\alpha\beta}) \in \mathbb{M}_{4\times 4} \,|\, X^{\alpha\beta} = X^{\beta\alpha} \text{ and } X^{0j} = 0 \,\}\,.$$

Then letting $\mathfrak{U} = (\mathfrak{U}^{\alpha\beta})$, $\mathfrak{U}$ takes values in $\mathbb{S}$. We employ the temporal gauge

$$A_0 = 0\,.$$

Therefore if we write the gauge potential A_i as a 3-tuple $A = (A_1, A_2, A_3)$ then the gauge potential A takes values in the space $\mathfrak{su}(2)^3$ which carries a norm

$$|A|^2 := \sum_{i=1}^{3} \langle A_i | A_i \rangle\,.$$

The above discussion shows that $\mathrm{W}^{k,p}_\delta(\mathbb{R}^3, \mathbb{S})$, $\mathrm{W}^{k,p}_\delta(\mathbb{R}^3)$ and $\mathrm{W}^{k,p}_\delta(\mathbb{R}^3, \mathfrak{su}(2)^3)$ are appropriate functions spaces for the static metric densities, dilaton fields, and gauge potentials, respectively.

In addition to being static, we will also assume that our fields are spherically symmetric. To define what we mean by spherical symmetry we first need to specify an action of $SO(3)$ on spacetime $\mathbb{R}^4$. We want $SO(3)$ to act on the hypersurfaces orthogonal to the timelike killing vector field ∂_0. So using the matrix representation of $SO(3)$ given by

$$SO(3) = \{\, a \in \mathbb{M}_{3\times 3} \,|\, a^t = a^{-1} \text{ and } \det(a) = 1 \,\}$$

we define a $SO(3)$ action on spacetime by

$$\Phi : SO(3) \times \mathbb{R}^4 \to \mathbb{R}^4 \ :\ (a, (x^0, \mathbf{x})) \to \Phi_a(x^0, \mathbf{x}) := (x^0, a\mathbf{x})$$

where we are treating $\mathbf{x}$ as a column vector and $a\mathbf{x}$ denotes matrix multiplication. We then get the induced action on functions via pullbacks. Therefore $SO(3)$ acts on the dilaton field $\psi(\mathbf{x})$ as follows

$$\Phi_a(\psi)(\mathbf{x}) := \psi(a^{\mathrm{t}}\mathbf{x})\,.$$

Lifting the $SO(3)$ action on spacetime to the tensor bundle, we get the following action on the metric densities

$$\Phi_a(\mathfrak{U})(\mathbf{x}) := \tilde{a}\mathfrak{U}(a^{\mathrm{t}}\mathbf{x})\tilde{a}^{\mathrm{t}}$$

where

$$\tilde{a} := \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\,.$$

Let $\widetilde{\mathrm{C}}_0^\infty(\mathbb{R}^3)$ denote the set of smooth $SO(3)$-invariant functions with compact support, i.e.

$$\widetilde{\mathrm{C}}_0^\infty(\mathbb{R}^3) := \{\, \psi \in \mathrm{C}_0^\infty(\mathbb{R}^3) \,|\, \psi = \Phi_a \psi \text{ for all } a \in SO(3) \,\} \,.$$

In other words, $\widetilde{\mathrm{C}}_0^\infty(\mathbb{R}^3)$ is the set of radial functions on $\mathbb{R}^3$. Similarly, define

$$\widetilde{\mathrm{C}}_0^\infty(\mathbb{R}^3, \mathbb{S}) := \{\, \mathfrak{U} \in \mathrm{C}_0^\infty(\mathbb{R}^3, \mathbb{S}) \,|\, \mathfrak{U} = \Phi_a \mathfrak{U} \text{ for all } a \in SO(3) \,\} \,.$$

In addition to being spherically symmetric, we will assume that our gauge potential is purely magnetic. Choosing an appropriate gauge, the gauge potential can then be written as [2]

$$A_i(\mathbf{x}) := u(\mathbf{x}) \epsilon_i{}^j{}_k x^k \tau_j$$

where $u(\mathbf{x}) = u(|\mathbf{x}|)$ and

$$\tau_1 = \frac{1}{2i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,,\; \tau_2 = \frac{1}{2i} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \,,\; \tau_3 = \frac{1}{2i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,,$$

is a basis for $\mathfrak{su}(2)$. Therefore we define the set of smooth static spherically symmetric purely magnetic gauge potentials with compact support by

$$\mathcal{A}_0^\infty := \{ A : \mathbb{R}^3 \to \mathfrak{su}(2)^3 \,|\, A_i(\mathbf{x}) = u(\mathbf{x}) \epsilon_i{}^j{}_k x^k \tau_j \text{ for some } u \in \widetilde{\mathrm{C}}_0^\infty(\mathbb{R}^3) \,\} \,.$$

Notice that every $A \in \mathcal{A}_0^\infty$ satisfies

$$\mathrm{div} A := \sum_{j=1}^{3} \partial_j A_j = 0 \,. \tag{4.1}$$

So then the spherically symmetric Sobolev spaces we consider are

$$\mathcal{D}_\delta^{k,p} := \overline{\widetilde{\mathrm{C}}_0^\infty(\mathbb{R}^3)} \subset \mathrm{W}_\delta^{k,p}(\mathbb{R}^3) \,, \tag{4.2}$$

$$\mathcal{U}_\delta^{k,p} := \overline{\widetilde{\mathrm{C}}_0^\infty(\mathbb{R}^3, \mathbb{S})} \subset \mathrm{W}_\delta^{k,p}(\mathbb{R}^3, \mathbb{S}) \,, \tag{4.3}$$

and

$$\mathcal{A}_\delta^{k,p} := \overline{\mathcal{A}_0^\infty} \subset \mathrm{W}_\delta^{k,p}(\mathbb{R}^3, \mathfrak{su}(2)^3) \,. \tag{4.4}$$

Because of (4.1) we have

$$\mathrm{div} A = 0 \quad \text{for all } A \in \mathcal{A}_\delta^{k,p} \tag{4.5}$$

by the density of $\mathcal{A}_0^\infty$ in $\mathcal{A}_\delta^{k,p}$ and the continuity of differentiation (see (3.1)).

Proposition 4.1. *For $-1 < \delta < 0$ the Laplacian $\Delta : \mathcal{D}_\delta^{k,p} \to \mathcal{D}_{\delta-2}^{k,p}$ is an isomorphism.*

Proof. Straightforward calculation shows that $\Delta(\widetilde{\mathrm{C}}_0^\infty(\mathbb{R}^3)) \subset \widetilde{\mathrm{C}}_0^\infty(\mathbb{R}^3)$. Using formula (3.3), it is not difficult to verify that if $\psi \in \widetilde{\mathrm{C}}_0^\infty(\mathbb{R}^3)$ then $\Phi_a(\Delta^{-1}\psi) = \Delta^{-1}\psi$ for all $a \in SO(3)$. The proposition then follows from these two results and theorem 3.4. ∎

The next proposition is proved in the same fashion.

Proposition 4.2. *For $-1 < \delta < 0$ the Laplacian $\Delta : \mathcal{U}^{k,p}_{\delta} \to \mathcal{U}^{k-2,p}_{\delta-2}$ is an isomorphism.*

We will often use the following notation

$$r := |\mathbf{x}| \quad \text{and} \quad (\cdot)' = \frac{d(\cdot)}{dr} \, .$$

Proposition 4.3. *For $-2 < \delta < 1$ the Laplacian $\Delta : \mathcal{A}^{2,p}_{\delta} \to \mathcal{A}^{0,p}_{\delta-2}$ is an isomorphism.*

Proof. By definition of $\mathcal{A}^{\infty}_{0}$, if $A \in \mathcal{A}^{\infty}_{0}$ then $A_i = u(r)\epsilon_i{}^j{}_k x^k \tau_j$ for some $u \in \widetilde{\mathrm{C}}^{\infty}_0(\mathbb{R}^3)$. So

$$\Delta A_i = \left(u''(r) + \frac{4}{r}u'(r) \right) \epsilon_i{}^j{}_k x^k \tau_j \tag{4.6}$$

and hence $\Delta(\mathcal{A}^{\infty}_0) \subset \mathcal{A}^{\infty}_0$. Therefore $\Delta : \mathcal{A}^{2,p}_{\delta} \to \mathcal{A}^{0,p}_{\delta-2}$ is continuous by the density of $\mathcal{A}^{\infty}_0$ and the continuity of $\Delta : \mathrm{W}^{2,p}_{\delta}(\mathbb{R}^3, \mathfrak{su}(2)^3) \to \mathrm{W}^{0,p}_{\delta-2}(\mathbb{R}^3, \mathfrak{su}(2)^3)$.

Let $\widetilde{\mathrm{C}}^{\infty}_0(\mathbb{R}^5)$ denote the set of smooth radial functions with compact support on $\mathbb{R}^5$. Then for $v \in \widetilde{\mathrm{C}}^{\infty}_0(\mathbb{R}^5)$ the action of the Laplacian is given by

$$\Delta v = v''(r) + \frac{4}{r}v'(r) \, . \tag{4.7}$$

Define $\mathrm{V}^{k,p}_{\delta} := \overline{\widetilde{\mathrm{C}}^{\infty}_0(\mathbb{R}^5)} \subset \mathrm{W}^{k,p}_{\delta}(\mathbb{R}^5)$. From (4.7) we have $\Delta(\widetilde{\mathrm{C}}^{\infty}_0(\mathbb{R}^5)) \subset \widetilde{\mathrm{C}}^{\infty}_0(\mathbb{R}^5)$ and hence it follows from the continuity of $\Delta : \mathrm{W}^{2,p}_{\delta}(\mathbb{R}^5) \to \mathrm{W}^{0,p}_{\delta-2}(\mathbb{R}^5)$ that $\Delta : \mathrm{V}^{2,p}_{\delta} \to \mathrm{V}^{0,p}_{\delta-2}$ is continuous. From the formula (3.3) it is easy to verify that if $v \in \widetilde{\mathrm{C}}^{\infty}_0(\mathbb{R}^5)$ then $\Delta^{-1}(v)$ is again a radial function and so $\Delta^{-1}(\mathrm{V}^{0,p}_{\delta-2}) \subset \mathrm{V}^{2,p}_{\delta}$ for $-3 < \delta < 0$ by theorem 3.4. This shows that

$$\Delta : \mathrm{V}^{2,p}_{\delta} \to \mathrm{V}^{0,p}_{\delta-2} \text{ is an isomorphism for } -3 < \delta < 0. \tag{4.8}$$

Any positive definite invariant product $\langle\cdot|\cdot\rangle$ on $\mathfrak{su}(2)$ is given by $\langle A|B\rangle = -2\alpha \mathrm{Tr}(AB)$ for all $A, B \in \mathfrak{su}(2)$ for some $\alpha > 0$. A short calculation shows that $|u(r)\epsilon_i{}^j{}_k x^k \tau_j|^2 = 2\alpha u(r)^2 r^2$. From this it is easy to establish that the map

$$T : \mathrm{V}^{k,p}_{\delta-1} \longrightarrow \mathcal{A}^{k,p}_{\delta};\ u(r) \longmapsto u(r)\epsilon_i{}^j{}_k x^k \tau_j \tag{4.9}$$

is an isomorphism. Moreover

$$T \circ \Delta = \Delta \circ T \tag{4.10}$$

by (4.6) and (4.7). The proof now follows as (4.8), (4.9), and (4.10) show that $\Delta : \mathcal{A}^{2,p}_{\delta} \to \mathcal{A}^{0,p}_{\delta-2}$ is an isomorphism for $-2 < \delta < 1$. ∎

5 Differentiability of the field equations

In this section we establish that the reduced field equations and the YMd equations define differentiable maps. In fact they define analytic maps. Before we proceed we first introduce some definitions.

Let $\mathcal{L}_k(B_1, B_2)$ denote the Banach space of k-linear and continuous maps from the Banach space B_1 into B_2 with norm

$$\|T\|_{\mathcal{L}_k(B_1,B_2)} := \sup\{\, \|T(x_1, \dots, x_k)\|_{B_2} \mid \sup\{\|x_1\|_{B_1}, \dots, \|x_k\|_{B_1}\} \le 1 \,\}\,.$$

Definition 5.1. *Let X_1 and X_2 be Banach spaces and let U be an open subset of X_1. A map $T : U \to V_2$ is said to be* analytic *if for each $x \in U$ there exists a $R > 0$ and a sequence $T_k \in \mathcal{L}(X_1, X_2)$ of k-linear symmetric maps such that*

$$\sum_{k=0}^{\infty} R^k \|T_k\|_{\mathcal{L}_k(X_1,X_2)} < \infty$$

and

$$T(y) = \sum_{k=0}^{\infty} T_k(y - x, \dots, y - x) \quad \textit{for all } y \textit{ with } \|y - x\|_{X_1} < R.$$

We use $C^\omega(U, X_2)$ to denote the set of all the analytic maps from U to X_2.

An open ball in a Banach space X will be denoted by

$$B_X(x; R) := \{\, y \in X \mid \|x - y\|_X < R\}$$

We then have the following useful proposition:

Proposition 5.2. *Suppose $u \in C^\omega(B_{\mathbb{R}^n}(0; R), \mathbb{R})$ satisfies $u(0) = 0$. Furthermore, suppose X is a commutative Banach algebra where C is any constant such that $\|xy\|_X \le C\|x\|_X\|y\|_X$ for all $x, y \in X$. Then the map*

$$\hat{u} : X^n \to X : (x_1, \dots, x_n) \mapsto \sum_{|I|=1}^{\infty} \frac{1}{I!} \left(\partial^I u(0)\right) x_1^{I_1} \dots x_n^{I_n}$$

is of class $C^\omega(B_X(0; \rho)^n, X)$ for $\rho = R/C$.

Note that

$$(\underset{o}{\bar{\mathfrak{g}}}{}^{\alpha\beta}) = \begin{pmatrix} -\lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.1}$$

so that

$$(\underset{o}{\bar{\mathfrak{g}}}{}^{\alpha\beta})\big|_{\lambda=0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2}$$

As in [11], we define for any weakly differentiable map u

$$u^{,\alpha} := \underset{o}{\bar{\mathfrak{g}}}^{\alpha\beta}\big|_{\lambda=0} u_{,\beta} = \begin{cases} \partial_\alpha u & \text{for } \alpha \neq 0 \\ 0 & \text{for } \alpha = 0 \end{cases} .$$

We now collect some results from [11] concerning the analyticity of various quantities involving the density $\mathfrak{U}$. We note that in [11] the propositions we quote were proved under the assumption that $p \geq 4$. However, using lemma 3.3 it is not difficult to verify that all the following proposition are valid for $p > 3$.

Proposition 5.3. [Proposition 3.10, [11]] *Suppose $p > 3$ and $-1 < \delta < 0$. Then for any $R > 0$ there exists a $\Lambda > 0$ such that the following maps are of class C^ω:*

$$(-\Lambda, \Lambda) \times B_{W^{2,p}_\delta(\mathbb{R}^3,\mathbb{S})}(0; R) \to W^{2,p}_\delta(\mathbb{R}^3, \mathbb{S}) \ : \ (\lambda, \mathfrak{U}) \mapsto (\bar{\mathfrak{g}}^{\alpha\beta} - \underset{o}{\bar{\mathfrak{g}}}^{\alpha\beta})$$
$$(-\Lambda, \Lambda) \times B_{W^{k,p}_\delta(\mathbb{R}^3,\mathbb{S})}(0; R) \to W^{2,p}_\delta(\mathbb{R}^3, \mathbb{S}) \ : \ (\lambda, \mathfrak{U}) \mapsto (\bar{\mathfrak{g}}_{\alpha\beta} - \underset{o}{\bar{\mathfrak{g}}}_{\alpha\beta})$$

and

$$(-\Lambda, \Lambda) \times B_{W^{2,p}_\delta(\mathbb{R}^3,\mathbb{S})}(0; R) \to W^{2,p}_\delta(\mathbb{R}^3) \ : \ (\lambda, \mathfrak{U}) \mapsto |\mathfrak{d}|^{q/2} - 1$$

for $q = -3, -2, -1, 1, 2$. Moreover, the following expansions are valid

$$|\mathfrak{d}| - 1 = -4\lambda\mathfrak{U}^{00} + O(\lambda^2) , \quad \sqrt{\mathfrak{d}} - 1 = -2\lambda\mathfrak{U}^{00} + O(\lambda^2) ,$$
$$\frac{1}{\sqrt{\mathfrak{d}}} - 1 = 2\lambda\mathfrak{U}^{00} + O(\lambda^2) , \quad (\bar{\mathfrak{g}}_{\alpha\beta} - \underset{o}{\bar{\mathfrak{g}}}_{\alpha\beta}) = -4\lambda(\delta^0_\alpha\delta^0_\beta)\mathfrak{U}^{00} + O(\lambda^2) .$$

Proposition 5.4. [Proposition 6.2, [11]] *Suppose $p > 3$ and $-1 < \delta < 0$. Then for any $R > 0$ there exists a $\Lambda > 0$ such that the Christofel symbols*

$$\Gamma^\alpha_{\beta\gamma} : (-\Lambda, \Lambda) \times B_{W^{2,p}_\delta(\mathbb{R}^3,\mathbb{S})}(0; R) \to W^{1,p}_{\delta-1}(\mathbb{R}^3)$$

are of class C^ω for all $\alpha, \beta, \gamma = 0, 1, 2, 3$. Moreover, the following expansion is valid

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}\big|_{\lambda=0} + O(\lambda)$$

where

$$\Gamma^\alpha_{\beta\gamma}\big|_{\lambda=0} = \begin{cases} \mathfrak{U}^{00}{}_{,\alpha} & \text{if } \beta = \gamma = 0 \text{ and } \alpha \neq 0 \\ 0 & \text{otherwise} \end{cases} .$$

Proposition 5.5. *Suppose $p > 3$ and $-1 < \delta < 0$. Then for any $R > 0$ there exists a Λ such that the map*

$$(E - \Delta) : (-\Lambda, \Lambda) \times B_{W^{2,p}_\delta(\mathbb{R}^3,\mathbb{S})}(0; R) \to W^{0,p}_{\delta-2}(\mathbb{R}^3, \mathbb{S}) \ : \ (\lambda, \mathfrak{U}) \mapsto (E^{\alpha\beta} - \Delta\mathfrak{U}^{\alpha\beta})$$

is of class C^ω where $E^{\alpha\beta}$ is defined by (2.18). Moreover,

$$D_2(E - \Delta)(0, \mathfrak{U}) \cdot \delta\mathfrak{U} = (\delta\mathfrak{U}^{00,\alpha}\mathfrak{U}^{00,\beta} + \delta\mathfrak{U}^{00,\beta}\mathfrak{U}^{00,\alpha} - \underset{o}{\bar{\mathfrak{g}}}^{\alpha\beta}\big|_{\lambda=0}\delta\mathfrak{U}^{00,\gamma}\mathfrak{U}^{00}{}_{,\gamma}) .$$

Proof. The proof of this proposition is contained in the proof of proposition 4.2 in [11]. □

Let

$$\Upsilon^2 := \bar{\mathfrak{g}}^{\alpha\beta}\left(\psi_{,\alpha\beta} - \Gamma^{\mu}_{\alpha\beta}\psi_{,\mu} - \frac{\kappa\ell_Y}{\ell_d}\frac{e^{2\kappa\psi}}{\sqrt{|\mathfrak{d}|}}\bar{\mathfrak{g}}^{\mu\nu}\langle F_{\alpha\mu}|F_{\beta\nu}\rangle\right), \tag{5.3}$$

and

$$\Upsilon^1_\beta := \bar{\mathfrak{g}}^{\alpha\nu}\left(F_{\alpha\beta,\nu} - \Gamma^{\mu}_{\alpha\nu}F_{\mu\beta} - \Gamma^{\mu}_{\beta\nu}F_{\alpha\mu} + 2\kappa\psi_{,\nu}F_{\alpha\beta} + [A_\nu, F_{\alpha\beta}]\right). \tag{5.4}$$

The YMd equations are then $\Upsilon^1 = (\Upsilon^1_\beta) = 0$ and $\Upsilon^2 = 0$.

We will split the gauge potential A and the dilaton field ψ as follows

$$A(x) = W(r) + Y(x) = W_i(r)dx^i + Y_i(x)dx^i \tag{5.5}$$

and

$$\psi(x) = \alpha(r) + \xi(x) \tag{5.6}$$

where

$$W_i(r) := \frac{w(r)-1}{r^2}\epsilon_i{}^j{}_k x^k \tau_j \tag{5.7}$$

and $\alpha(r)$ are to be considered as *fixed*. Under the splitting (5.5), the gauge potential decomposes as

$$F_{\alpha\beta} = F^W_{\alpha\beta} + F^Y_{\alpha\beta} + [Y_\alpha, W_\beta] + [W_\alpha, Y_\beta] \tag{5.8}$$

where

$$F^W_{\alpha\beta} := \partial_\alpha W_\beta - \partial_\beta W_\alpha + [W_\alpha, W_\beta] \tag{5.9}$$

and

$$F^Y_{\alpha\beta} := \partial_\alpha Y_\beta - \partial_\beta Y_\alpha + [Y_\alpha, Y_\beta]. \tag{5.10}$$

A short calculation shows that non-zero components of $F^W_{\alpha\beta}$ are

$$F^W_{ij} = \epsilon_{ijk}\left[\frac{w'(r)}{r}\left(\delta^{kl} - \frac{x^k x^l}{r^2}\right) + \frac{w^2-1}{r^4}x^k x^l\right]\tau_l. \tag{5.11}$$

Proposition 5.6. *Suppose $w(r) \in \mathrm{C}^2((0,\infty))$ satisfies*

1. $w(r) - 1 \in O(r^2)$, $w'(r) \in O(r)$ *as* $r \to 0$,
2. $w(r) + 1 \in O(r^{-\sigma})$ *or* $w(r) - 1 \in O(r^{-\sigma})$ *as* $r \to \infty$,
3. $w'(r) \in O(r^{-(\sigma+1)})$ *and* $w''(r) \in O(r^{-(\sigma+2)})$ *as* $r \to \infty$.

Then $W_\alpha \in \mathcal{A}^{2,p}_{\delta_1}$ *for any* $\delta_1 > -1$, $p > 1$ *and* $F^W_{\alpha\beta} \in W^{2,p}_{\delta_2}(\mathbb{R}^3, \mathfrak{su}(2))$ *for any* $\delta_2 > -(2+\sigma)$, $p > 1$.

Proof. Follows directly from the formulas (5.7), (5.11) and the definition of the weighted Sobolev spaces. □

Remark 5.7. *For the remainder of this article, we will assume*

- $w(r)$ *satisfies the hypotheses of proposition 5.6 for some* $0 < \sigma \leq 1$,
- $\alpha \in \mathcal{D}^{2,p}_\delta$ *for* $-1 > \delta > 0$.

Proposition 5.8. *Suppose* $-(1+\sigma) > \delta > -1$ *and* $p > 3$. *Then for any* $R > 0$ *and* $\alpha, \beta, \gamma = 0, 1, 2, 3$ *the following maps are* C^ω

$$B_{W^{2,p}_\delta(\mathbb{R}^3,\mathfrak{su}(2)^3)}(0;R) \to W^{1,p}_{\delta-1}(\mathbb{R}^3, \mathfrak{su}(2)) \; : \; (Y_j) \mapsto F_{\alpha\beta}\,,$$

and

$$B_{W^{2,p}_\delta(\mathbb{R}^3,\mathfrak{su}(2)^3)}(0;R) \to W^{0,p}_{\delta-2}(\mathbb{R}^3, \mathfrak{su}(2)) \; : \; (Y_j) \mapsto [A_\alpha, F_{\beta\gamma}]\,,$$

whre A_α *and* $F_{\alpha\beta}$ *are given by the formula* (5.5) *and* (5.8), *respectively.*

Proof. The proof is a direct consequence of lemma 3.3 and proposition 5.6. □

Proposition 5.9. *Suppose* $p > 3$ *and* $-1 < \delta_1 < 0$ *and* $-(1+\sigma) < \delta_2 < -1$. *Then for any* $R > 0$ *there exists a* $\Lambda > 0$ *such that the map*

$$\begin{aligned} \Upsilon : (-\Lambda, \Lambda) \times B_{W^{2,p}_{\delta_1}(\mathbb{R}^3,\mathbb{S})}(0;R) \times B_{W^{2,p}_{\delta_2}(\mathbb{R}^3,\mathfrak{su}(2)^3)}(0;R) \times B_{W^{2,p}_{\delta_1}(\mathbb{R}^3)}(0;R) \\ \longrightarrow W^{0,p}_{\delta_1-2}(\mathbb{R}^3) \times W^{0,p}_{\delta_2-2}(\mathbb{R}^3, \mathfrak{su}(2)^3) \; : \; (\lambda, \mathfrak{U}, Y, \xi) \longmapsto (\Upsilon^1, \Upsilon^2) \end{aligned}$$

is of class C^ω.

Proof. Follows easily from lemma 3.3 and propositions 5.2, 5.3, 5.4, and 5.8. □

Proposition 5.10. *Suppose* $p > 3$ *and* $-1 < \delta_1 < 0$ *and* $-(1+\sigma) < \delta_2 < -1$. *Then for any* $R > 0$ *there exists* $\Lambda > 0$ *and* $\epsilon > 0$ *such that the maps*

$$\begin{aligned} T : (-\Lambda, \Lambda) \times B_{W^{2,p}_{\delta_1}(\mathbb{R}^3,\mathbb{S})}(0;R) \times B_{W^{2,p}_{\delta_2}(\mathbb{R}^3,\mathfrak{su}(2)^3)}(0;R) \times B_{W^{2,p}_{\delta_1}(\mathbb{R}^3)}(0;R) \\ \longrightarrow W^{1,p}_{\delta_1-(2+\epsilon)}(\mathbb{R}^3, \mathbb{S}) \; : \; (\lambda, \mathfrak{U}, Y, \xi) \longmapsto (T^{\alpha\beta}) \end{aligned}$$

and

$$\begin{aligned} \mathcal{T} : (-\Lambda, \Lambda) \times B_{W^{2,p}_{\delta_1}(\mathbb{R}^3,\mathbb{S})}(0;R) \times B_{W^{2,p}_{\delta_2}(\mathbb{R}^3,\mathfrak{su}(2)^3)}(0;R) \times B_{W^{2,p}_{\delta_1}(\mathbb{R}^3)}(0;R) \\ \longrightarrow W^{1,p}_{\delta_1-(2+\epsilon)}(\mathbb{R}^3, \mathbb{S}) \; : \; (\lambda, \mathfrak{U}, Y, \xi) \longmapsto (\mathcal{T}^{\alpha\beta}) \end{aligned}$$

are of class C^ω.

Proof. Follows easily from lemma 3.3 and propositions 5.2, 5.3, and 5.8. □

We now prove spherically symmetric versions of propositions 5.5, 5.9 and 5.10.

Proposition 5.11. *Suppose $p > 3$ and $-1 < \delta < 0$. Then for any $R > 0$ there exists a Λ such that the map*

$$(E - \Delta) : (-\Lambda, \Lambda) \times B_{\mathcal{U}^{2,p}_{\delta}}(0; R) \to \mathfrak{U}^{0,p}_{\delta-2} \; : \; (\lambda, \mathfrak{U}) \mapsto (E^{\alpha\beta} - \Delta\mathfrak{U}^{\alpha\beta})$$

is of class C^ω. Moreover,

$$D_2(E - \Delta)(\lambda, \mathfrak{U}) \cdot \delta\mathfrak{U} = (\delta\mathfrak{U}^{00,\alpha}\mathfrak{U}^{00,\beta} + \delta\mathfrak{U}^{00,\beta}\mathfrak{U}^{00,\alpha} - \underset{o}{\bar{\mathfrak{g}}}^{\alpha\beta}\big|_{\lambda=0}\delta\mathfrak{U}^{00,\gamma}\mathfrak{U}^{00}{}_{,\gamma}) \, .$$

Proof. Given R, let Λ be determined as in proposition 5.5. By straightforward calculation it can be shown that if $\mathfrak{U} \in \widetilde{\mathrm{C}}^\infty_0(\mathbb{R}^3, \mathbb{S}) \cap B_{\mathrm{W}^{2,p}_\delta(\mathbb{R}^3,\mathbb{S})}(0; R)$ then $(E - \Delta)(\mathfrak{U}) \in \widetilde{\mathrm{C}}^\infty_0(\mathbb{R}^3, \mathbb{S})$. Consequently $(E - \Delta) \left(\widetilde{\mathrm{C}}^\infty_0(\mathbb{R}^3, \mathbb{S}) \cap B_{\mathrm{W}^{2,p}_\delta(\mathbb{R}^3,\mathbb{S})}(0; R)\right) \subset \widetilde{\mathrm{C}}^\infty_0(\mathbb{R}^3, \mathbb{S})$. Therefore $(E - \Delta)\left(B_{\mathcal{U}^{2,p}_\delta}(0; R)\right) \subset \mathcal{U}^{0,p}_{\delta-2}$ by the density of $\widetilde{\mathrm{C}}^\infty_0(\mathbb{R}^3, \mathbb{S})$ in $\mathcal{U}^{2,p}_\eta$ for $\eta \in \mathbb{R}$, and continuity of the map $(E - \Delta)$ by proposition 5.5. The proposition now follows from proposition 5.5. □

Proposition 5.12. *Suppose $p > 3$ and $-1 < \delta_1 < 0$ and $-(1 + \sigma) < \delta_2 < -1$. Then for any $R > 0$ there exists a $\Lambda > 0$ such that the map*

$$\begin{aligned} \Upsilon : (-\Lambda, \Lambda) \times B_{\mathcal{U}^{2,p}_{\delta_1}}(0; R) \times B_{\mathcal{A}^{2,p}_{\delta_2}}(0; R) \times B_{\mathcal{D}^{2,p}_{\delta_1}}(0; R) \\ \longrightarrow \mathcal{A}^{0,p}_{\delta_2-2} \times \mathcal{D}^{0,p}_{\delta_1-2} \; : \; (\lambda, \mathfrak{U}, Y, \xi) \longmapsto (\Upsilon^1, \Upsilon^2) \end{aligned}$$

is of class C^ω.

Proof. As in the proof of proposition 5.11, straightforward calculation shows that if $\mathfrak{U} \in \widetilde{\mathrm{C}}^\infty_0(\mathbb{R}^3, \mathbb{S}) \cap B_{\mathrm{W}^{2,p}_\delta(\mathbb{R}^3,\mathbb{S})}(0; R)$, $Y \in \mathcal{A}^\infty_0 \cap B_{\mathrm{W}^{2,p}_{\delta_2}(\mathbb{R}^3,\mathfrak{su}(2)^3)}(0; R)$ and $\xi \in \widetilde{\mathrm{C}}^\infty_0(\mathbb{R}^3) \cap B_{\mathrm{W}^{2,p}_{\delta_1}(\mathbb{R}^3)}(0; R)$ then $\Upsilon(Y, \alpha) \in \widetilde{\mathrm{C}}^\infty_0(\mathbb{R}^3) \cap \mathrm{C}^2 \times \mathcal{A}^\infty_0 \cap \mathrm{C}^2$. We then argue in the same manner as proposition 5.11. □

Proposition 5.13. *Suppose $p > 3$ and $-1 < \delta_1 < 0$ and $-(1 + \sigma) < \delta_2 < -1$. Then for any $R > 0$ there exists $\Lambda > 0$ and $\epsilon > 0$ such that the maps*

$$\begin{aligned} T : (-\Lambda, \Lambda) \times B_{\mathcal{U}^{2,p}_{\delta_1}}(0; R) \times B_{\mathcal{A}^{2,p}_{\delta_2}}(0; R) \times B_{\mathcal{D}^{2,p}_{\delta_1}}(0; R) \\ \longrightarrow \mathcal{U}^{1,p}_{\delta_1-(2+\epsilon)} \; : \; (\lambda, \mathfrak{U}, Y, \xi) \longmapsto (T^{\alpha\beta}) \end{aligned}$$

and

$$\begin{aligned} \mathcal{T} : (-\Lambda, \Lambda) \times B_{\mathcal{U}^{2,p}_{\delta_1}}(0; R) \times B_{\mathcal{A}^{2,p}_{\delta_2}}(0; R) \times B_{\mathcal{D}^{2,p}_{\delta_1}}(0; R) \\ \longrightarrow \mathcal{U}^{1,p}_{\delta_1-(2+\epsilon)} \; : \; (\lambda, \mathfrak{U}, Y, \xi) \longmapsto (\mathcal{T}^{\alpha\beta}) \end{aligned}$$

are of class C^ω.

Proof. See the proofs of propositions 5.11 and 5.12. □

From (3.2) and propositions 4.2, 5.11 and 5.13 we get the following:

Proposition 5.14. *Suppose $-1 < \delta_1 < 0$, $-(1+\sigma) < \delta_2 < -1$ and $p > 3$. Then for any $R > 0$ there exists a $\Lambda > 0$ such that*

$$\begin{aligned}\Xi : (-\Lambda, \Lambda) \times B_{\mathcal{U}^{2,p}_{\delta_1}}(0; R) \times B_{\mathcal{A}^{2,p}_{\delta_2}}(0; R) \times B_{\mathcal{D}^{2,p}_{\delta_1}}(0; R) &\longrightarrow \mathcal{U}^{2,p}_{\delta 1} \\ : \ (\lambda, \mathfrak{U}, Y, \xi) &\mapsto \left(\mathfrak{U}^{\alpha\beta} - \Delta^{-1}\left\{\mathcal{T}^{\alpha\beta} - (E^{\alpha\beta} - \Delta\mathfrak{U}^{\alpha\beta})\right\}\right)\end{aligned}$$

is of class C^ω.

From the definition of Ξ it is clear that the reduced field equations (2.16) are equivalent to $\Xi = 0$.

6 Solving the reduced field equations

We now employ the same method as in [11] to find solutions to the reduced field equations. Namely, we first solve the reduced equations for $\lambda = 0$, and then use an implicit function argument to show that there exist a solution for λ small enough.

6.1 $\lambda = 0$

Assume $-1 < \delta_1 < 0$, $-(1+\sigma) < \delta_2 < -1$, $p > 3$ and for fixed $R > 0$ let $\Lambda > 0$ be as in proposition 5.14. From the expansions in proposition 5.3 and (2.9) we see that

$$E^{\alpha\beta}\big|_{\lambda=0} = \Delta\mathfrak{U}^{\alpha\beta} + \begin{cases} -\mathfrak{U}^{00,\alpha}\mathfrak{U}^{00,\beta} + \frac{1}{2}\delta^{\alpha\beta}|\mathrm{grad}\mathfrak{U}^{00}|^2 & \text{if } \alpha \neq 0, \beta \neq 0 \\ 0 & \text{otherwise} \end{cases}, \tag{6.1}$$

and

$$\begin{aligned}\mathcal{T}^{\alpha\beta} =& 2\pi G\ell_d \left(\left(\underset{o}{\bar{\mathfrak{g}}}^{\alpha\mu}\underset{o}{\bar{\mathfrak{g}}}^{\beta\nu}\right)\Big|_{\lambda=0}\psi_{,\mu}\psi_{,\nu} - \tfrac{1}{2}\left(\underset{o}{\bar{\mathfrak{g}}}^{\alpha\beta}\underset{o}{\bar{\mathfrak{g}}}^{\mu\nu}\right)\Big|_{\lambda=0}\psi_{,\mu}\psi_{,\nu}\right) + \\ & 4\pi G\ell_Y e^{2\kappa\psi}\left(\left(\underset{o}{\bar{\mathfrak{g}}}^{\alpha\mu}\underset{o}{\bar{\mathfrak{g}}}^{\beta\nu}\underset{o}{\bar{\mathfrak{g}}}^{\sigma\tau}\right)\Big|_{\lambda=0}\langle F_{\mu\sigma}|F_{\nu\tau}\rangle - \tfrac{1}{4}\left(\underset{o}{\bar{\mathfrak{g}}}^{\mu\nu}\underset{o}{\bar{\mathfrak{g}}}^{\sigma\tau}\underset{o}{\bar{\mathfrak{g}}}^{\alpha\beta}\right)\Big|_{\lambda=0}\langle F_{\mu\sigma}|F_{\nu\tau}\rangle\right).\end{aligned}$$

So then

$$\Xi(0, \mathfrak{U}, Y, \xi) = 0 \quad \Longleftrightarrow$$

$$\mathfrak{U}^{\alpha\beta} = \begin{cases} \Delta^{-1}\left(\mathcal{T}^{\alpha\beta} + \mathfrak{U}^{00,\alpha}\mathfrak{U}^{00,\beta} - \frac{1}{2}\delta^{\alpha\beta}|\mathrm{grad}\mathfrak{U}^{00}|^2\right) & \text{if } \alpha \neq 0, \beta \neq 0 \\ \Delta^{-1}\mathcal{T}^{\alpha\beta} & \text{if } \alpha = 0, \beta \neq 0 \text{ or } \alpha \neq 0, \beta = 0 \\ 0 & \text{if } \alpha = \beta = 0 \end{cases}.$$

Consequently

$$\mathfrak{U}^{\alpha\beta} = \begin{cases} \Delta^{-1}\mathcal{T}^{\alpha\beta} & \text{if } \alpha \neq 0, \beta \neq 0 \\ 0 & \text{if } \alpha = \beta = 0 \end{cases} \tag{6.2}$$

solves $\Xi(0, \mathfrak{U}, Y, \xi) = 0$ for any $Y \in B_{\mathcal{A}^{2,p}_{\delta_2}}(0; R)$ and $\xi \in B_{\mathcal{D}^{2,p}_{\delta_1}}(0; R)$.

6.2 $\lambda \neq 0$

Proposition 6.1. *Suppose* $-1 < \delta_1 < 0$, $-(1+\sigma) < \delta_2 < -1$, *and* $p > 3$. *Then there exists a* $\Lambda > 0$, $\epsilon > 0$ *and a* C^∞ *map*

$$\hat{\mathfrak{U}} : (-\Lambda, \Lambda) \times B_{\mathcal{A}^{2,p}_{\delta_2}}(0;\epsilon) \times B_{\mathcal{D}^{2,p}_{\delta_1}}(0;\epsilon) \to \mathcal{U}^{2,p}_{\delta_1} \;:\; (\lambda, Y, \xi) \to \hat{\mathfrak{U}}(\lambda, Y, \xi) = (\hat{\mathfrak{U}}^{\alpha\beta}(\lambda, Y, \xi))$$

such that $\Xi(\lambda, \hat{\mathfrak{U}}(\lambda, Y, \xi), Y, \xi) = 0$ *for all* $(\lambda, Y, \xi) \in (-\Lambda, \Lambda) \times B_{\mathcal{A}^{2,p}_{\delta_2}}(0;\epsilon) \times B_{\mathcal{D}^{2,p}_{\delta_1}}(0;\epsilon)$. *Moreover,* $\hat{\mathfrak{U}}$ *satisfies* $\hat{\mathfrak{U}}^{00}(0,0,0) = 0$, $D_2\hat{\mathfrak{U}}^{00}(0,0,0) = 0$, *and* $D_3\hat{\mathfrak{U}}^{00}(0,0,0) = 0$.

Proof. Fix $R > 0$ and let $\Lambda > 0$ be chosen so that the maps Ξ, $E - \Delta$ and $\mathcal{T}$ are of class C^ω which we can do by propositions 5.11, 5.13, and 5.14. Then we can solve $\Xi(0, \mathfrak{U}, 0, 0) = 0$ by (6.2). Let $\mathfrak{U}_b$ denote the solution. Note that $\mathfrak{U}_b^{00} = 0$ again by (6.2). So $\mathrm{D}_2(E - \Delta)(\lambda, \mathfrak{U}_b) = 0$ by proposition (5.11). From the expansions in proposition 5.3, and formula (2.9) it follows that $\mathrm{D}_2\mathcal{T}(0, \mathfrak{U}_b, 0, 0) = 0$. Therefore from the definition of Ξ it is clear that

$$\mathrm{D}_2\Xi(0, \mathfrak{U}_b, 0, 0) = 1\!\!1_{\,\mathcal{U}^{2,p}_{\delta_1}}\,, \tag{6.3}$$

and hence by the implicit function theorem there exists a $\bar{\Lambda} > 0$, $\epsilon > 0$, and a C^∞ map

$$\hat{\mathfrak{U}} : (-\bar{\Lambda}, \bar{\Lambda}) \times B_{\mathcal{A}^{2,p}_{\delta_2}}(0;\epsilon) \times B_{\mathcal{D}^{2,p}_{\delta_1}}(0;\epsilon) \to \mathcal{U}^{2,p}_{\delta_1} \;:\; (\lambda, Y, \xi) \to \hat{\mathfrak{U}}(\lambda, Y, \xi) = (\hat{\mathfrak{U}}^{\alpha\beta}(\lambda, Y, \xi))$$

such that

$$\Xi(\lambda, \hat{\mathfrak{U}}(\lambda, Y, \xi), Y, \xi) = 0 \tag{6.4}$$

for all $(\lambda, Y, \xi) \in (-\bar{\Lambda}, \bar{\Lambda}) \times B_{\mathcal{A}^{2,p}_{\delta_2}}(0;\epsilon) \times B_{\mathcal{D}^{2,p}_{\delta_1}}(0;\epsilon)$. Differentiating (6.4) with respect to Y and using (6.3) we find

$$\mathrm{D}_2\hat{\mathfrak{U}}^{00}(0,0,0) = -D_3\Xi^{00}(0, \mathfrak{U}_b, 0, 0)\,. \tag{6.5}$$

But

$$D_3\Xi^{00}(\lambda, \mathfrak{U}, Y, \xi)\cdot\delta Y = \bigg(-4\pi G\Delta^{-1}\bigg\{\frac{\ell_Y}{\sqrt{|\mathfrak{d}|}}e^{-2\kappa\psi}\Big(\bar{\mathfrak{g}}^{\alpha\mu}\bar{\mathfrak{g}}^{\beta\nu}\bar{\mathfrak{g}}^{\sigma\tau}\Big[\langle\delta F_{\mu\sigma}|F_{\nu\tau}\rangle + \langle F_{\mu\sigma}|\delta F_{\nu\tau}\rangle\Big] - \tfrac{1}{2}\langle\delta F_{\sigma\mu}|F_{\nu\tau}\rangle\bar{\mathfrak{g}}^{\mu\nu}\bar{\mathfrak{g}}^{\sigma\tau}\bar{\mathfrak{g}}^{\alpha\beta}\Big)\bigg\}\bigg)$$

where

$$\delta F_{\alpha\beta} = \partial_\alpha\delta Y_\beta - \partial_\beta\delta Y_\alpha + [\delta Y_\alpha, Y_\beta] + [Y_\alpha, \delta Y_\beta] + [\delta Y_\alpha, W_\beta] + [W_\alpha, \delta Y_\beta] \tag{6.6}$$

and $F_{\alpha\beta}$ is given by the formula (5.8). Setting $\lambda = 0$ we get, by (2.9), (5.2), and the expansions of proposition 5.3, that $D_2\Xi^{00}(0, \mathfrak{U}_b, 0, 0) = 0$. Therefore $\mathrm{D}_2\mathfrak{U}^{00}(0,0,0) = 0$ by (6.5). Similar calculations show that $\mathrm{D}_3\mathfrak{U}^{00}(0,0,0) = 0$. $\square$

7 Solving the YMd equations

Suppose $-1 < \delta_1 < 0$, $-(1+\sigma) < \delta_2 < -1$, $p > 3$ and let Λ, ϵ and $\hat{\mathfrak{U}}$ be as in proposition 6.1. Then by the results of propositions 5.12 and 6.1 the map

$$\hat{\Upsilon} : (-\Lambda, \Lambda) \times B_{\mathcal{A}^{2,p}_{\delta_2}}(0;\epsilon) \times B_{\mathcal{D}^{2,p}_{\delta_1}}(0;\epsilon) \to \mathcal{D}^{0,p}_{\delta_1-2} \times \mathcal{A}^{0,p}_{\delta_2-2} \tag{7.1}$$

defined by

$$\hat{\Upsilon}(\lambda, Y, \xi) := \Upsilon(\lambda, \hat{\mathfrak{U}}(\lambda, Y, \xi), Y, \xi) \tag{7.2}$$

is C^∞.

Define

$$\hat{\Gamma}^\alpha_{\beta\gamma}(\lambda, Y, \xi) := \Gamma^\alpha_{\beta\gamma}(\lambda, \hat{\mathfrak{U}}(\lambda, Y, \xi))\,.$$

Then (2.24), (2.9), (5.2), the expansions of proposition 5.3, and proposition 6.1 show that

$$\hat{\Gamma}^\alpha_{\beta\gamma}(0,0,0) = 0\,,\ \mathrm{D}_2\hat{\Gamma}^\alpha_{\beta\gamma}(0,0,0) = 0 \ \text{ and } \mathrm{D}_1\hat{\Gamma}^\alpha_{\beta\gamma}(0,0,0) = 0\,.$$

Using this result along with (2.9), (4.5), (5.2), and the expansions of proposition 5.3, we find after straightforward calculation that

$$\hat{\Upsilon}^2(0,0,0) = \Delta\alpha - \frac{\kappa\ell_Y}{\ell_d} e^{2\kappa\alpha}\delta^{ij}\delta^{kl}\langle F^W_{ik} | F^W_{jl}\rangle \tag{7.3}$$

$$\hat{\Upsilon}^1(0,0,0) = \big(\delta^{ik}\big(\partial_k F^W_{ij} + 2\kappa F^W_{ij}\partial_k\alpha + [W_k, F^W_{jk}]\big)\big) \tag{7.4}$$

and

$$\begin{aligned}\mathrm{D}_2\hat{\Upsilon}^1(0,0,0)\cdot\delta Y &= \big(\Delta\delta Y_j + \delta^{ik}\big([\delta Y_i, \partial_k W_j] \\ &\quad + [W_i, \partial_k \delta Y_j] + 2\kappa\delta F_{ij}\partial_k\alpha + [\delta Y_k, F^W_{ij}] + [W_k, \delta F_{ij}]\big)\big)\,, \end{aligned}\tag{7.5}$$

$$\mathrm{D}_3\hat{\Upsilon}^1(0,0,0)\cdot\delta\xi = \big(2\kappa\delta^{ik}F^W_{ij}\partial_k\delta\xi\big)\,, \tag{7.6}$$

$$\mathrm{D}_2\hat{\Upsilon}^2(0,0,0)\cdot\delta Y = -\frac{2\kappa\ell_Y}{\ell_d}\delta^{ij}\delta^{kl}\langle F^W_{ik} | \delta F_{jl}\rangle\,, \tag{7.7}$$

$$\mathrm{D}_3\hat{\Upsilon}^2(0,0,0)\cdot\delta\xi = \Delta\delta\xi - \frac{2\kappa\ell_Y}{\ell_d}\delta\xi e^{2\kappa\alpha}\delta^{ij}\delta^{kl}\langle F^W_{ik} | F^W_{jl}\rangle\,, \tag{7.8}$$

where F^W_{ij} and δF_{ij} are given by the formulas (5.9) and (6.6), respectively. Observe that (7.3) and (7.4) are precisely the flat space static spherically symmetric Yang-Mills-dilaton equations.

We can collect (7.5)-(7.8) into a single matrix expression

$$\mathcal{K}\begin{pmatrix}\delta Y \\ \delta\xi\end{pmatrix} = \begin{pmatrix}\Delta & 0 \\ 0 & \Delta\end{pmatrix}\begin{pmatrix}\delta Y \\ \delta\xi\end{pmatrix} + \begin{pmatrix}K_{11} & K_{12} \\ K_{21} & K_{22}\end{pmatrix}\begin{pmatrix}\delta Y \\ \delta\xi\end{pmatrix} \tag{7.9}$$

where

$$
\begin{aligned}
K_{11}\cdot\delta Y &:= \big(\delta^{ik}\big([\delta Y_i,\partial_k W_j]+[W_i,\partial_k\delta Y_j] \\
&\qquad + 2\kappa\delta F_{ij}\partial_k\alpha + [\delta Y_k, F^W_{ij}] + [W_k,\delta F_{ij}]\big)\big)\,, && (7.10)\\
K_{12}\cdot\delta\xi &:= \big(2\kappa\delta^{ik}F^W_{ij}\partial_k\delta\xi\big)\,, && (7.11)\\
K_{21}\cdot\delta Y &:= -\frac{2\kappa\ell_Y}{\ell_d}\delta^{ij}\delta^{kl}\langle F^W_{ik}|\delta F_{jl}\rangle\,, && (7.12)\\
K_{22}\cdot\delta\xi &:= -\frac{2\kappa\ell_Y}{\ell_d}\delta\xi e^{2\kappa\alpha}\delta^{ij}\delta^{kl}\langle F^W_{ik}|F^W_{jl}\rangle\,. && (7.13)
\end{aligned}
$$

To solve the YMd equations we again employ an implicit function technique where we assume that (W,α) is a solution to the YMd equations, i.e.

$$\hat{\Upsilon}(0,0,0)=0 \tag{7.14}$$

In order to use the implicit function theorem we need prove that

$$\mathcal{K}\;:\;\mathcal{A}^{2,p}_{\delta_2}\times\mathcal{D}^{2,p}_{\delta_1}\longrightarrow\mathcal{A}^{0,p}_{\delta_2-2}\times\mathcal{D}^{0,p}_{\delta_1-2}$$

is an isomorphism. As the next result shows, it will be enough to establish that $\ker\mathcal{K}=\{0\}$. The difficulty in proving that $\mathcal{K}$ is an isomorphism lies with the fact that its spectrum contains both strictly negative and positive components. Therefore the fact that $\ker\mathcal{K}=\{0\}$ cannot be proved by an integration by parts argument. It is worth noting that the negative part of the spectrum accounts for the well known instability of the Yang-Mills-dilaton solutions.

Proposition 7.1. $\ker\mathcal{K}=\{0\}$ *if and only if* $\mathcal{K}$ *is an isomorphism.*

Proof. Let

$$K=\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}\,.$$

Since $-(1+\sigma)<\delta_2<-1$ and $-1<\delta_1<0$ there exists and $\epsilon>0$ such that $-(1+\sigma)<\delta_2<-(1+\epsilon)$ and $-1<\delta_1<-\epsilon$. Therefore $K(\mathcal{A}^{2,p}_{\delta_2}\times\mathcal{D}^{2,p}_{\delta_1})\subset\mathcal{A}^{1,p}_{\delta_2-(2+\epsilon)}\times\mathcal{D}^{1,p}_{\delta_1-(1+\epsilon)}$ by lemma 3.3. But the embedding $\mathcal{A}^{1,p}_{\delta_2-(2+\epsilon)}\times\mathcal{D}^{1,p}_{\delta_1-(2+\epsilon)}\to\mathcal{A}^{0,p}_{\delta_2-2}\times\mathcal{D}^{0,p}_{\delta_1-2}$ is compact by lemma 3.5 and hence $K:\mathcal{A}^{2,p}_{\delta_2}\times\mathcal{D}^{2,p}_{\delta_1}\to\mathcal{A}^{0,p}_{\delta_2-2}\times\mathcal{D}^{0,p}_{\delta_1-2}$ is compact. As $\Delta\oplus\Delta:\mathcal{A}^{2,p}_{\delta_2}\times\mathcal{D}^{2,p}_{\delta_1}\to\mathcal{A}^{0,p}_{\delta_2-2}\times\mathcal{D}^{0,p}_{\delta_1-2}$ is an isomorphism by propositions 4.1 and 4.3 it follows by compactness of K that $\mathrm{Index}\,(\Delta\oplus\Delta+K)=0$ and the proof is complete. □

Proposition 7.2. *If* (W,α) *is a non-trivial* C^2 *solution to the the flat space YMd equations* (7.14) *where* $\alpha\in\mathcal{D}^{2,p}_{\delta_1}$ *and* W *satisfies the hypothesis of proposition 5.6, then*

$$\mathcal{K}\;:\;\mathcal{A}^{2,p}_{\delta_2}\times\mathcal{D}^{2,p}_{\delta_1}\to\mathcal{A}^{0,p}_{\delta_2-2}\times\mathcal{D}^{0,p}_{\delta_1-2}$$

is an isomorphism.

Proof. By assumption (W, α) is a classical solutions to the YMd equations. Using (5.7) the YMd equations become

$$w'' = -\alpha' w' + \frac{(w^2-1)w}{r^2}\,, \tag{7.15}$$

$$(r^2\alpha')' = e^{\alpha}\left(w'^2 + \frac{(w^2-1)^2}{2r^2}\right)\,. \tag{7.16}$$

We have absorbed the coupling constants by a suitable scaling of r and α.

Suppose $(\delta Y, \delta\xi) \in \mathcal{A}^{2,p}_{\delta_2} \times \mathcal{D}^{2,p}_{\delta_1}$ is a solution to

$$\mathcal{K}\begin{pmatrix}\delta Y\\ \delta\xi\end{pmatrix} = 0\,. \tag{7.17}$$

Note that this equation is nothing more that the linearized Yang-Mills-dilaton (lYMd) equations. Now, $\mathcal{K}$ is uniformly elliptic and has coefficients in C^2 since W and ξ are C^2 by assumption. Therefore by elliptic regularity, see [9] theorem 9.19 or [8] theorem 3.6, $\delta Y \in \mathrm{C}^2 \cap \mathcal{A}^{2,p}_{\delta_2}$ and $\delta\xi \in \mathrm{C}^2 \cap \mathcal{D}^{2,p}_{\delta_1}$. Letting

$$\phi = \delta\xi \quad \text{and} \quad \delta Y_i = \frac{v(r)}{r^2}\epsilon_i{}^j{}_k x^k \tau_j$$

the lYMd equations (7.17) can be written as

$$v'' + \phi' w' + \alpha' v' - \frac{(3w^2-1)}{r^2}v = 0\,, \tag{7.18}$$

$$(r^2\phi')' - e^{\alpha}\left(w'^2 + \frac{(w^2-1)^2}{2r^2}\right)\phi - 2e^{\alpha}\left(w'v' + \frac{(w^2-1)}{r^2}wv\right) = 0\,. \tag{7.19}$$

Lemma 7.3.

$$\phi(r) := \frac{r}{2}\alpha' - 1 \quad v(r) := \frac{r}{2}w'(r) \tag{7.20}$$

is a solution to the lYMd equations (7.18) *and* (7.19).

Proof. The lemma is proved by substitution of $\phi(r) = \frac{r}{2}\alpha' - 1$ and $v(r) = \frac{r}{2}w'(r)$ into (7.18)-(7.19) and repeated use of the YMd equations (7.15)-(7.16). □

The proof of this lemma may be understood heuristically as follows. Since w and α are solutions to the YMd equations (7.15)-(7.16), it can be verified that

$$w_\beta(r) := w(e^{\beta/2}r) \quad \alpha_\beta := \alpha(e^{\beta/2}r) - \beta \tag{7.21}$$

also solve the YMd equations for any $\beta \in \mathbb{R}$. Thus (w_β, α_β) is a one-parameter family of solutions that passes through our fixed solution (w, α) at $\beta = 0$. Therefore one expects that

$$\frac{d}{d\beta}\Big|_{\beta=0}(w_\beta, \alpha_\beta) = \left(\frac{r}{2}w'(r), \frac{r}{2}\alpha'(r) - 1\right) \tag{7.22}$$

would be a solution to the lYMd equations and indeed this is what we find.

Now,

$$\delta Y(x) \in \mathrm{o}(r^{\delta_2})\,, \quad \partial_j \delta Y(x) \in \mathrm{o}(r^{\delta_2-1})$$
$$\alpha(x),\, \delta\xi(x) \in \mathrm{o}(r^{\delta_1})\,, \quad \partial_j\alpha(x),\, \partial_j\delta\xi(x) \in \mathrm{o}(r^{\delta_1-1}),$$

as $r \to \infty$ by [1] theorem 1.2 (iv). Therefore, the above results and our assumptions on $w(r)$ imply that

$$w(r)-1 \text{ or } w(r)+1 \in \mathrm{o}(r^{\delta_2+1})\,, \quad v(r) \in \mathrm{o}(r^{\delta_2+1})\,, \quad w'(r)\,,\, v'(r) \in \mathrm{o}(r^{\delta_2})\,, \tag{7.23}$$
$$\alpha(r)\,,\, \phi(r) \in \mathrm{o}(r^{\delta_1})\,, \quad \alpha'(r)\,,\, \phi'(r) \in \mathrm{o}(r^{\delta_1-1})\,, \tag{7.24}$$

as $r \to \infty$.

Lemma 7.4. *Let $\eta = \min\{-(\delta_2+1), -\delta_1\}$. Then there is at most one non-trivial solution (ϕ, v) to the lYMd equations* (7.18)-(7.19) *on* $(0,\infty)$ *satisfying*

$$v(r) \in o(r^{-\eta}) \quad \textit{and} \quad \phi'(r), v'(r) \in o(r^{-(\eta+1)}) \quad \textit{as } r \to \infty. \tag{7.25}$$

Proof. Suppose $w'(r) = 0$ on an interval $[r_1, r_2]$ where $0 < r_1 < r_2$. Then $w = 0$ or $w = 1$ by (7.15). If $w = 0$ on $[r_1, r_2]$ then by uniqueness of solutions to differential equations if follows from (7.15)-(7.16) that $w = 0$ on $(0,\infty)$. But then $|w(r)-1| = 1$ and $|w(r)+1| = 1$. So the fall off condition (7.23) is violated and hence w cannot be identically zero on an interval $[r_1, r_2]$. On the other hand, if $w = \pm 1$ on $[r_1, r_2]$ then similar arguments show that $w = \pm 1$ on $(0,\infty)$. Thus $W = 0$ by (5.7) and hence $\Delta\alpha = 0$ by (7.3) and (7.14). As Δ is an isomorphism we must have $\alpha = 0$. But this contradicts the assumption that (W, α) was non-trivial solution to (7.14). Thus w' cannot be identically zero on any interval. Moreover, by the Cauchy-Kowalevski w will be analytic on $(0,\infty)$. So the set $\{\, w'(r) = 0 \,|\, r \in (0,\infty)\}$ cannot have any limit points and hence is discrete.

Let

$$P(r) = {w'}^2 + \frac{(w^2-1)^2}{2r^2}\,.$$

Suppose $r_0 \in (0,\infty)$ and $P(r_0) = 0$. Then $w'(r_0) = 0$ and $w(r_0) = \pm 1$. Again appealing to the uniqueness of solutions to differential equations we can conclude that $w = \pm 1$ on $(0,\infty)$. But from the arguments above we know that this cannot happen and therefore $P(r) > 0$ for all $r \in (0,\infty)$.

Let (ϕ_1, v_1) and (ϕ_2, v_2) be two non-trivial linearly independent solutions to the lYMd equations (7.18)-(7.19). Suppose that there exists a smooth function $f(r)$ on (r_1, r_2) , $0 < r_1 < r_2$, such that $f(r) \neq 0$ for any $r \in (r_1, r_2)$ and

$$f(r)\begin{pmatrix} v_1(r) \\ v_1'(r) \end{pmatrix} = \begin{pmatrix} v_1(r) \\ v_2'(r) \end{pmatrix}\,. \tag{7.26}$$

Then $v_2' = (fv_1)' = f'v_1' + fv_1' = f'v_1 + v_2'$ and hence $f'v_1 = 0$ on (r_1, r_2). Therefore $f' = 0$ or $v_1 = v_2 = 0$ on (r_1, r_2).

Suppose $f' = 0$ on (r_1, r_2). Then $f'(r) = f_0$ on (r_1, r_2) for some constant $f_0 \neq 0$. Therefore $v_2 = f_0 v_1$ on (r_1, r_2). The lYMd equations (7.18)-(7.19) then imply that

$$(r^2\phi_1')' - e^\alpha P\phi_1 - 2e^\alpha\left(w'v_1' + \frac{(w^2-1)}{r^2}wv_1\right) = 0 \tag{7.27}$$

$$v_1'' + \phi_1' w' + \alpha' v_1' - \frac{(3w^2-1)}{r^2}v_1 = 0\,, \tag{7.28}$$

$$(r^2\phi_2')' - e^\alpha P\phi_2 - f_0 2e^\alpha\left(w'v_1' + \frac{(w^2-1)}{r^2}wv_1\right) = 0 \tag{7.29}$$

$$v_1'' + \frac{1}{f_0}\phi_1' w' + \alpha' v_1' - \frac{(3w^2-1)}{r^2}v_1 = 0\,. \tag{7.30}$$

Subtracting (7.28) and (7.30) we get $w'(\phi_1' - \phi_2'/f_0) = 0$. As w' is zero for only a discrete set of points we must have $(\phi_1 - \phi_2/f_0)' = 0$ on (r_1, r_2). So there exists a constant b_0 such that $\phi_2 = f_0\phi_1 + b_0$ and it follows from (7.27)-(7.30) that $e^\alpha P b_0/f_0 = 0$. As $e^\alpha P > 0$ we must have $b_0 = 0$. Thus $\phi_2 = f_0\phi_1$. We have established that $\phi_2(r) = f_0\phi_1(r)$ and $v_2(r) = f_0 v_1(r)$ for all $r \in (r_1, r_2)$ and hence it remains true for all $r \in (0, \infty)$ by uniqueness of solutions to differential equations. But this contradicts the assumption that (v_1, ϕ_1) and (v_2, ϕ_2) were linearly independent. Therefore we conclude that f' cannot be identically zero on (r_1, r_2).

So suppose now that $v_1 = v_2 = 0$ on (r_1, r_2). Then the lYMd equation (7.18) implies that $\phi_1' w' = \phi_2' w'$ on (r_1, r_2). Again using the fact that $w' = 0$ only on a discrete set of point we find that $\phi_1' = \phi_2' = 0$ on (r_1, r_2). Therefore there exists constants b_1, b_2 such that $\phi_1 = b_1$ and $\phi_2 = b_2$ on (r_1, r_2). Using the other lYMd equation (7.19) we see that $e^\alpha P b_1 = e^\alpha P b_2 = 0$ since $e^\alpha P > 0$. So $b_1 = b_2 = 0$ and we arrive at $\phi_1 = \phi_2 = 0$ on (r_1, r_2). Again by uniqueness of solutions to differential equations we conclude that $v_1 = v_2 = \phi_1 = \phi_2 = 0$ on $(0, \infty)$. But this contradicts the assumption that the solutions (v_1, ϕ_1) and (v_2, ϕ_2) were not trivial. Thus we cannot have $v_1 = v_2 = 0$ on (r_1, r_2). This shows that for any interval (r_1, r_2) there does not exist a non-vanishing function $f(r)$ such that (7.26) holds. In other words the two vectors

$$\begin{pmatrix} v_1 \\ v_1' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_2 \\ v_2' \end{pmatrix} \tag{7.31}$$

are linearly independent every where on $(0, \infty)$ except for perhaps a discrete set of points.

Note from the definition of η that $\eta \in (0, 1)$ since $-(1+\sigma) < \delta_2 < -1$ and $-1 < \delta_1 < 0$. Also from the definition of η and (7.23) and (7.24) we see that $w - 1$ or $w + 1 \in \mathrm{o}(r^{-\eta})$, w' $\alpha' \in \mathrm{o}(r^{-\eta})$ as $r \to \infty$. By assumption $v \in \mathrm{o}(r^{-\eta})$, and ϕ', $v' \in \mathrm{o}(r^{-\eta})$ as $r \to \infty$. Therefore letting

$$h := -\phi' w' - \alpha' v' + \frac{3(w^2-1)}{r^2}v$$

we see that

$$h \in \mathrm{o}(r^{-2(1+\eta)}) \quad \text{as } r \to \infty. \tag{7.32}$$

Let

$$v = b_1 v_1 + b_2 v_2$$

where b_1, b_2 are constants to be chosen later. Using h, we see that v satisfies the differential equation

$$v'' - \frac{2}{r^2} = h\,. \tag{7.33}$$

Consider the initial value problem

$$q'' - \frac{2}{r^2} = 0 \quad ; \quad q(r_0) = 1\,,\; q'(r_0) = \frac{2}{r_0}\,. \tag{7.34}$$

Choose r_0 large enough so that

$$|h(r)| \leq \frac{1}{2r_0^2} \qquad \text{for all } r \geq r_0 \tag{7.35}$$

which we can do by (7.32). By (7.31) and increasing r_0 by a small amount if necessary we can always choose b_1 and b_2 so that

$$v(r_0) = \frac{3}{2} \quad \text{and} \quad v'(r_0) = \frac{4}{r_0}\,. \tag{7.36}$$

From (7.34) and (7.36) we get

$$(v-q)'' = \frac{2}{r^2}(v-q) + h \tag{7.37}$$

and

$$(v-q)'(r_0) > 0 \quad \text{and} \quad (v-q)(r_0) = \frac{1}{2}\,. \tag{7.38}$$

So

$$(v-q)''(r_0) = \frac{1}{r_0^2} + h(r_0) \geq \frac{1}{r_0^2} - \frac{1}{2r_0^2} > 0 \tag{7.39}$$

by (7.35), (7.37), and (7.38). Therefore there exists an $\epsilon > 0$ such that $(v-q)''(r) > 0$ for $r_0 \leq r \leq r_0 + \epsilon$. This implies that $(v-q)'(r) > 0$ for $r_0 \leq r \leq r_0 + \epsilon$ by (7.38) and hence $(v-q)(r) \geq 1/2$ for $r_0 \leq r \leq r_0 + \epsilon$ again by (7.38). Suppose r_1 is the first $r > r_0$ for which $(v-q)'(r) = 0$. Then using the same arguments used to derive (7.39) we find that $(v-q)''(r_1) > 0$ which contradicts $(v-q)'(r_1) = 0$. Therefore we conclude that $(v-q)'(r) > 0$ for all $r \geq r_0$ which implies that $(v-q)(r) \geq 1/2$ for all $r \geq r_0$. Solving the initial value problem (7.34) we find that $q(r) = r^2/(2r_0^2)$ and hence $v(r) \geq 1/2 + r^2/(2r_0^2)$ for all $r \geq r_0$. But this contradicts the fall off condition $v(r) \in \mathrm{o}(r^{-\eta})$. So we conclude that there cannot exist two linearly independent solutions to the lYMd equations (7.18)-(7.19) satisfying the conditions (7.25). □

It is clear from (7.23) and (7.24) that the lYMd solution (7.20) satisfies (7.25). This must be the unique lYMd solution satisfying (7.25) and hence $\ker \mathcal{K} = \{0\}$ as the solution (7.20) does not lie in $\mathcal{A}^{2,p}_{\delta_2} \times \mathcal{D}^{2,p}_{\delta_1}$. This proves, by proposition 7.1, that $\mathcal{K}$ is an isomorphism. □

We are now ready to solve the YMd equations.

Proposition 7.5. *Suppose $-1 < \delta_1 < 0$, $-(1+\sigma) < \delta_2 < -1$, $p > 3$ and let Λ and ϵ be as in* (7.1)*. If (W, α) is a non-trivial C^2 solution to the the flat space YMd equations* (7.14) *where $\alpha \in \mathcal{D}^{2,p}_{\delta_1}$ and W satisfies the hypothesis of proposition 5.6, then there exists $\hat{\Lambda} \in (0, \Lambda)$ and two C^∞ maps*

$$\hat{Y} : (-\hat{\Lambda}, \hat{\Lambda}) \to B_{\mathcal{A}^{2,p}_{\delta_2}}(0; \epsilon) \quad \text{and} \quad \hat{\xi} : (-\hat{\Lambda}, \hat{\Lambda}) \to B_{\mathcal{D}^{2,p},\delta_1}(0; \epsilon)$$

such that $\hat{Y}(0) = 0$, $\hat{\xi}(0) = 0$ and $\hat{\Upsilon}(\lambda, \hat{Y}(\lambda), \hat{\xi}(\lambda)) = 0$ for all $\lambda \in (-\hat{\Lambda}, \hat{\Lambda})$.

Proof. Because $\mathcal{K} : \mathcal{A}^{2,p}_{\delta_2} \times \mathcal{D}^{2,p}_{\delta_1} \to \mathcal{A}^{0,p}_{\delta_2-2} \times \mathcal{D}^{0,p}_{\delta_1-2}$ is an isomorphism by propositions 7.2 we can apply the implicit functions theorem to get the desired result. $\square$

8 Solving the EYMd field equations

By the propositions 6.1 and 7.5 we can solve the reduced field equations (2.17) and the YMd equations (2.22)-(2.23) provided we have a classical solutions of the flat space YMd equations (7.15)-(7.16). Using the following result of Heilig [11], we will see that this solution will actually be a solution to the full EYMd equations.

Proposition 8.1. [propostion 6.1, [11]] *Suppose $-1 < \delta < 0$, $p > 3$, and $\Lambda > 0$. Furthermore, suppose*

$$T : [0, \Lambda] \to \mathrm{W}^{0,p}_{\delta-2}(\mathbb{R}^3, \mathbb{S}^3) \cap \mathrm{C}^1(\mathbb{R}^3, \mathbb{S}^3) \;:\; \lambda \mapsto (T^{\alpha\beta}_\lambda)$$

and

$$U\text{: }[0,\Lambda] \to \mathrm{W}^{2,p}_{\delta}(\mathbb{R}^3, \mathbb{S}^3) \;:\; \lambda \mapsto (\mathfrak{U}^{\alpha\beta}_\lambda)$$

are two continuous maps such that for every $\lambda \in [0, \Lambda]$: $(\lambda, \mathfrak{U}^{\alpha\beta}_\lambda, T^{\alpha\beta}_\lambda)$ is a solution to the reduced field equations 2.16, $\nabla_\beta T^{\alpha\beta}_\lambda = 0$, and $\partial_\gamma T^{\alpha\beta}_\lambda \in B_{\mathrm{W}^{0,p}_{\delta-2}(\mathbb{R}^3)}(0, R)$ for some $R > 0$ independent of λ and α, β, γ. Then there exists a constant $\hat{\Lambda} \in (0, \Lambda]$ such that $\partial_\alpha \mathfrak{U}^{\alpha\beta}_\lambda = 0$ for all $\lambda \in [0, \hat{\Lambda}]$.

Proposition 8.2. *Suppose $-1 < \delta_1 < 0$, $-(1+\sigma) < \delta_2 < 1$, $p > 3$, (W, α) is a non-trivial C^2 solution to the the flat space YMd equations* (7.14) *where $\alpha \in \mathcal{D}^{2,p}_{\delta_1}$ and W satisfies the hypothesis of proposition 5.6. Then there exist a $\Lambda > 0$ and C^∞ maps $\mathfrak{U} : [-\Lambda, \Lambda] \to \mathcal{U}^{2,p}_{\delta_2} \;:\; \lambda \mapsto (\mathfrak{U}^{\alpha\beta}_\lambda)$, $Y : [-\Lambda, \Lambda] \to \mathcal{A}^{2,p}_{\delta_2} \;:\; \lambda \mapsto (Y^\lambda_\alpha)$, and $\xi : [-\Lambda, \Lambda] \to \mathcal{D}^{2,p}_{\delta_1} \;:\; \lambda \mapsto \xi^\lambda$ such that $(Y^0, \xi^0) = (0, 0)$ and for any $\lambda \in (0, \Lambda]$ $(\lambda, \mathfrak{U}^{\alpha\beta}_\lambda, A^\lambda = W + Y^\lambda_\alpha, \psi^\lambda = \alpha + \xi^\lambda)$ is a C^2 solution to the EYMd equations* (2.6), (2.22), *and* (2.23)*.*

Proof. Since (W, α) is a non-trivial C^2 solution to the the flat space YMd equations (7.14) where $\alpha \in \mathcal{D}^{2,p}_{\delta_1}$ and W satisfies the hypothesis of proposition 5.6, we find by propositions 6.1 and 7.5 that there exists a $\Lambda > 0$ and C^∞ maps $\mathfrak{U} : [-\Lambda, \Lambda] \to \mathcal{U}^{2,p}_{\delta_2}$, $:\; \lambda \mapsto (\mathfrak{U}^{\alpha\beta}_\lambda)$, $Y : [-\Lambda, \Lambda] \to \mathcal{A}^{2,p}_{\delta_2} \;:\; \lambda \mapsto (Y^\lambda_\alpha)$, and $\xi : [-\Lambda, \Lambda] \to \mathcal{D}^{2,p}_{\delta_1} \;:\; \lambda \mapsto \xi^\lambda_\alpha$, such that $(Y^0, \xi^0) = (0, 0)$, and

$$\Xi(\lambda, \mathfrak{U}(\lambda), Y(\lambda), \xi(\lambda)) = 0\,, \quad \Upsilon(\lambda, \mathfrak{U}(\lambda), Y(\lambda), \xi(\lambda)) = 0$$

for all $\lambda \in (-\Lambda, \Lambda)$. To reduce notation, we will often write $\mathfrak{U}$, Y and ξ instead of $\mathfrak{U}_\lambda$, Y^λ, and ξ^λ.

Lemma 8.3. *There exists a $\Lambda^* \in (0, \Lambda]$ such that $A^\lambda = W + Y^\lambda, \psi^\lambda = \alpha + \xi^\lambda \in \mathrm{C}^2$ for all $\lambda \in (-\Lambda^*, \Lambda^*)$.*

Proof. Let $\mathrm{B}_R \subset \mathbb{R}^3$ be an open ball of radius R centered at the origin. Then ψ, $\psi_{,\alpha}$, $\mathfrak{U}^{\alpha\beta}$, $\mathfrak{U}^{\alpha\beta}{}_{,\mu}$, A_α, $A_{\alpha,\beta} \in \mathrm{W}^{1,p}(\mathrm{B}_R)$, where recall that $A = W + Y$ and $\psi = \alpha + \xi$. As $\mathrm{W}^{1,p}(\mathrm{B}_R)$ is a Banach algebra, we have

$$f := \Gamma^\mu_{\alpha\beta}\psi_{,\mu}\bar{\mathfrak{g}}^{\alpha\beta} - \frac{\kappa\ell_Y}{\ell_d}\bar{\mathfrak{g}}^{\alpha\mu}\langle F_{\alpha\mu}|F_{\alpha\beta}\rangle \in \mathrm{W}^{1,p}(\mathrm{B}_R)$$
$$h = (h_j) := \left(\bar{\mathfrak{g}}^{\alpha\nu}\left(\Gamma^\mu_{\alpha\nu}F_{\mu j} + \Gamma^\mu_{j\nu}F_{\alpha\mu} - 2\kappa\psi_{,\nu}F_{\alpha j} - [A_\nu, F_{\alpha,j}]\right)\right) \in \mathrm{W}^{1,p}(\mathrm{B}_R, \mathbb{R}^3)\,,$$
$$\bar{\mathfrak{g}}^{ij} = \delta^{ij} + 4\lambda^2\mathfrak{U}^{ij} \in W^{1,p}(\mathrm{B}_R)$$
$$Q^{ik} = \left(Q^{ikl}{}_j\right) := \left((\delta^{ik} + 4\lambda^2\mathfrak{U}^{ik})\delta^l_j - 4\lambda^2\mathfrak{U}^{lk}\delta^i_j\right) \in \mathrm{W}^{1,p}(\mathrm{B}_R, \mathbb{M}_{3\times 3})$$

and hence $f, h_j, \bar{\mathfrak{g}}^{ij}, Q^{ikl}{}_j \in \mathrm{C}^{0,1-3/p}(\mathrm{B}_R)$ by the Sobolev embedding theorem. Notice that YMd equations $\Upsilon(\lambda, \mathfrak{U}(\lambda), Y(\lambda), \xi(\lambda)) = 0$ can be written as

$$\bar{\mathfrak{g}}^{ij}\partial^2_{x^ix^j}\psi = f \quad \text{and} \quad Q^{ijl}{}_j\partial^2_{x^ix^j}A_l = h_j\,.$$

It is obvious that $\mathrm{W}^{1,p}_{\delta_1}(\mathbb{R}^3) \subset \mathrm{W}^{1,p}(\mathbb{R}^3)$ and hence $(-\Lambda, \Lambda) \to \mathrm{C}^0(\mathbb{R}^3):\ \ \lambda \mapsto U^{\alpha\beta}_\lambda$ is continuous by the Sobolev embedding theorem. Therefore there exists a $\Lambda^* \in (0, \Lambda)$ such that the operators $\bar{\mathfrak{g}}^{ij}\partial^2_{x^ix^j}$ and $Q^{ij}\partial^2_{x^ix^j}$ are uniformly elliptic on $\mathbb{R}^3$ for all $\lambda \in [-\Lambda^*, \Lambda^*]$. By elliptic regularity, $A^\lambda_j = W_j + Y^\lambda_j$ and $\psi^\lambda = \alpha + \psi^\lambda$ are in $\mathrm{C}^2(\mathrm{B}_R)$ for all $\lambda \in [-\Lambda^*, \Lambda^*]$. As Λ^* is independent of R the result follows. □

It follows immediately from equation (2.26), proposition 5.10, and the above lemma that the hypotheses of proposition 8.1 are satisfied. Therefore we conclude that there exist a constant $\hat{\Lambda} \in (0, \Lambda^*]$ such that

$$\partial_\alpha\mathfrak{U}^{\alpha\beta}_\lambda = 0 \tag{8.1}$$

for all $\lambda \in [0, \hat{\Lambda}]$. This implies that the full EYMd equations are equivalent to $\Xi = 0$ and $\Upsilon = 0$ and hence $(\lambda, \mathfrak{U}_\lambda, A^\lambda = W + Y^\lambda, \psi^\lambda = \alpha + \xi^\lambda)$ satisfy the EYMd equations for all $\lambda \in (0, \hat{\Lambda}]$.

Using (8.1), the reduced field equations $\Xi = 0$ can be written as

$$\bar{\mathfrak{g}}^{ij}\partial^2_{x^ix^j}\mathfrak{U}^{\alpha\beta} = H^{\alpha\beta}$$

where $H^{\alpha\beta} = -A^{\alpha\beta} - B^{\alpha\beta} - C^{\alpha\beta} + 4\pi G|\mathfrak{d}|T^{\alpha\beta}$. As in lemma 8.3, it can be shown that for any $\lambda \in (0, \hat{\Lambda})$ and $R > 0$ that $H^{\alpha\beta} \in \mathrm{C}^{0,1-3/p}(\mathrm{B}_R)$. Since $\hat{\Lambda} \leq \Lambda^*$, $\bar{\mathfrak{g}}^{ij}\partial^2_{x^ix^j}$ is uniformly elliptic on $\mathbb{R}^3$. Therefore by elliptic regularity we have $\mathfrak{U}^{\alpha\beta} \in \mathrm{C}^2$. □

9 Existence

Proposition 8.2 from the previous section shows that if we have a non-trivial C^2-solution $\left(W_i = \frac{w(r)-1}{r^2}\epsilon_i{}^j{}_k x^k\tau_j, \alpha(r)\right)$ to the YMd equation such that $\alpha \in \mathcal{D}^{2,p}_{\delta_1}$ where $-1 < \delta_1 < 0$, and $w(r)$ satisfies the hypotheses of proposition 5.6 for some $1 \geq \sigma > 0$ then we will automatically get C^2 static spherically symmetric EYMd solutions. The next theorem provides the existence of an infinite number of solutions to the YMd equations. However, it is not clear from the theorem if any of the solutions satisfy our assumptions Therefore we cannot, without more work, immediately conclude existence of EYMd solutions.

Theorem 9.1. [theorem 1, [10]] *There exists a sequence $n = 1, 2, 3, \ldots$ of C^∞ solutions $(w_n(r), \alpha_n(r))$ to the flat YMd equations* (7.15)-(7.16) *defined on* $(0, \infty)$ *such that w_n has precisely n local maxima and minima and $\lim_{r\to\infty} w_n(r) = (-1)^n$.*

Remark 9.2. *It is also established in [10] that the solutions $(w_n(r), \alpha_n(r))$ from theorem 9.1 satisfy the following*

1. $\lim_{r\to\infty} \alpha_n(r) = const,$
2. $|w| \leq 1$, $w_n' \in \mathrm{o}(r^{-1})$ *and* $\alpha_n' \in \mathrm{O}(r^{-2})$ *as* $r \to \infty$,
3. $w_n(r)$ *and* $\alpha_n(r)$ *are analytic in a neighborhood of* $r = 0$ *and* $w_n - 1 \in \mathrm{O}(r^2)$ *as* $r \to 0$,
4. w_n' *is either strictly positive or negative for r large enough.*

By using the scaling transformation (7.21)*, we can assume* $\lim_{r\to\infty} \alpha_n(r) = 0$.

The next proposition will be used to show that for all $n = 1, 2, 3, \ldots$ the YMd solutions from theorem 9.1 satify the appropriate conditions which allows us to apply proposition 8.2.

Proposition 9.3. *Suppose $(w(r), \alpha(r))$ is a solution to the flat YMd equations* (7.15)-(7.16) *defined on* $(0, \infty)$ *that satisfies $|w(r)| < 1$ for all $r \in (0, \infty)$, $\lim_{r\searrow 0} w'(r) = 0$, $\lim_{r\to\infty} w(r) = 1$ or $\lim_{r\to\infty} w(r) = -1$, $\lim_{r\to\infty} \alpha(r) = 0$, $w' \in \mathrm{o}(1/r)$ and $\alpha' \in \mathrm{O}(r^{-2})$ as $r \to \infty$. Furthermore, suppose that there exist a $R > 0$ such that $w'(r), w(r) > 0$ or $w'(r), w(r) < 0$ for all $r \geq R$. Then $w'' \in \mathrm{O}(r^{-5/2})$, $w' \in \mathrm{O}(r^{-3/2})$, $w - 1$ or $w + 1 \in \mathrm{O}(r^{-1/2})$, $\alpha'' \in \mathrm{O}(r^{-3})$, and $\alpha \in \mathrm{O}(r^{-1})$ as $r \to \infty$.*

Proof. Define

$$u := 1 - w^2$$

and

$$Z_\pm := \frac{1 - w^2}{r} - 2ww' \pm \frac{w'}{r^{1/2}} \, .$$

Note that

$$0 < u(r) \leq 1 \quad \forall\, r \in (0, \infty)$$

as $|w| < 1$ on $(0, \infty)$. Also note that Z can be written as

$$Z_\pm = \frac{u}{r} + 2u' \pm \frac{w'}{r^{1/2}} \, .$$

Lemma 9.4. *If $w'(r), w(r) > 0$ ($w'(r), w(r) < 0$) for $r \geq R$ and there exist a $R^* \geq R$ such that $Z_+(r) < 0$ ($Z_-(r) < 0$) for all $r \geq R^*$ then $w' \in \mathrm{O}(r^{-3/2})$ as $r \to \infty$.*

Proof. We only proof the case where $w'(r)$ and $w(r)$ are both positive for sufficiently large r. The other case follows using similar arguments. Since $w(r) > 0$ for $r \geq R$, $Z_+(r) < 0$ for $r \geq R^*$ implies that

$$\frac{1}{r} \leq -2\frac{u'}{u} \quad \forall\, r \geq R^*$$

as $u > 0$. Integrating this expression between R^* and r yields

$$\ln\left(\frac{r}{R^*}\right) < \ln\left(\frac{u(R^*)^2}{u(r)^2}\right) ,$$

or equivalently

$$u(r) < \frac{C}{\sqrt{r}} \quad \forall\, r \geq R^* \tag{9.1}$$

where $C = u(R^*)\sqrt{R^*}$. Note that (7.15) can be written $(e^\alpha w')' = -r^{-2}wu$. Then for $r \geq R^*$, integration yields

$$\begin{aligned} e^{\alpha(r)}w'(r) &= \int_r^\infty \frac{wu}{\rho^2}d\rho && \text{(since } \lim_{r\to\infty} e^{\alpha(r)}w'(r) = 0) \\ &\leq \int_r^\infty \frac{C}{\rho^{3/2}}d\rho && \text{(by (9.1) and } |w| \leq 1) \\ &= \frac{2C}{3}\frac{1}{r^{2/3}} . \end{aligned}$$

The result then follows since $w'(r) > 0$ for $r \geq R$ and $\lim_{r\to\infty} \alpha(r) = 0$. □

Lemma 9.5. *If $w'(r), w(r) > 0$ ($w'(r), w(r) < 0$) for $r \geq R$ and there exists a $R^* \geq R$ such that $Z_+(r) > 0$ ($Z_-(r) > 0$)for all $r \geq R^*$ then for any $\epsilon \in (0, 1)$ $w' \in \mathrm{O}(r^{-2\epsilon})$. In particular for $\epsilon = 3/4$, $w' \in \mathrm{O}(r^{-3/2})$.*

Proof. Again, we only proof the case where $w'(r)$ and $w(r)$ are both positive for sufficiently large r, with the other cases following from similar arguments. Since $\lim_{r\to\infty} w(r) = 1$ there exists a $\tilde{R} \geq R^*$ such that $w(r) > 0$ for all $r \geq \tilde{R}$. Therefore, $Z_+(r) > 0$ for $r \geq R^*$ implies that

$$\frac{w(1-w^2)}{r^2} - \frac{2w^2w'}{r} + \frac{ww'}{r^{3/2}} > 0 \quad \forall\, r \geq \tilde{R}$$

as $w' > 0$ for all $r \geq R$. It then follows from (7.15) that

$$-w'' - \alpha'w' > \frac{2w^2w'}{r} - \frac{ww'}{2r^{3/2}} \quad \forall\, r \geq \tilde{R} .$$

Fix $\epsilon \in (0, 1)$. As $\lim_{r\to\infty} w(r) = 1$, there exists a $R_\epsilon \geq \tilde{R}$ such that $w(r) \geq \sqrt{\epsilon}$ for all $r \geq R_\epsilon$. Thus

$$-w'' - \alpha'w' > \frac{2\epsilon}{r}w' - \frac{w'}{2r^{3/2}} \quad \forall\, r \geq R_\epsilon . \tag{9.2}$$

Note that in deriving this inequality we have also used $|w| \leq 1$. Dividing (9.2) by w' yields

$$-\frac{w''}{w'} > -\alpha' + \frac{2\epsilon}{r} - \frac{1}{2r^{3/2}} \quad \forall\, r \geq R_\epsilon\,.$$

Integrating gives

$$\ln\left(\frac{w'(R_\epsilon)}{w'(r)}\right) > \alpha(r) - \alpha(R_\epsilon) + \ln\left(\left(\frac{r}{R_\epsilon}\right)^{2\epsilon}\right) - \frac{1}{\sqrt{R_\epsilon}} \quad \forall\, r \geq R_\epsilon\,,$$

and hence

$$w'(r) < \left(\frac{w'(R_\epsilon)e^{\alpha(r)}e^{2R_\epsilon^{-1/2}}}{e^{\alpha(R_\epsilon)}}\right)\frac{1}{r^{2\epsilon}} \quad \forall\, r \geq R_\epsilon\,.$$

The proof then follows as $\lim_{r\to\infty}\alpha(r) = 0$ and $w'(r) > 0$ for all $r \geq R_\epsilon$. □

Lemma 9.6.

$$w' \in \mathrm{O}(r^{-2/3})$$

Proof. We need to consider two cases, namely $w'(r), w(r) > 0$ and $w'(r), w(r) < 0$ for $r \geq R$. We will prove the lemma assuming that $w'(r), w(r) > 0$ for $r \geq R$ with the other case following from similar arguments. We may assume that there exists a sequence $\{r_n\}_{n=1}^\infty$ such that $R \leq r_1 < r_2 < r_3 < \ldots$, $\lim_{n\to\infty} r_n = \infty$, and $Z_+(r_n) = 0 \quad n = 1,2,3,\ldots$ because otherwise we are done by lemmas 9.4 and 9.5. From (7.15), it is easy to verify that $u = 1 - w^2$ satisfies

$$u'' = \frac{2w^2}{r^2}u - 2|w'|^2 + 2w\alpha'w'\,. \tag{9.3}$$

Define

$$f(r) := 2|w'|^2 + \frac{wu}{r^2} + \left(\frac{3}{2r^{3/2}} + \frac{\alpha'}{r^{1/2}} - 2w\alpha'\right)w'\,. \tag{9.4}$$

Since $\alpha' \in \mathrm{O}(r^{-2})$, $|w| \leq 1$, and $w'(r) > 0$ for all $r \geq R$, there exists a $\tilde{R} \geq R$ such that

$$f(r) > 0 \quad \forall\, r \geq \tilde{R}. \tag{9.5}$$

Choose $m \in \mathbb{N}$ large enough so that

$$r_m \geq \tilde{R}\,. \tag{9.6}$$

By definition of the r_n we have

$$Z_+(r_m) = \frac{u(r_m)}{r_m} + u'(r_m) + \frac{w'(r_m)}{r_m} = 0\,. \tag{9.7}$$

Consider the following initial value problem

$$v'' = \frac{2}{r^2}v + \frac{3w'}{2r^{3/2}} + \frac{\alpha' w'}{r^{1/2}} + \frac{wu}{r^2}\,, \tag{9.8}$$

$$v(r_m) = u(r_m) \quad v'(r_m) = u'(r_m)\,. \tag{9.9}$$

From (9.3) and (9.8) we see that

$$(v-u)'' = \frac{2}{r^2}(v - w^2 w) + f(r)\,. \tag{9.10}$$

Then $|w| \leq 1$, (9.4), (9.5), (9.9), and (9.10) imply that $(v-u)''(r_m) > 0$. Therefore there exists an $\epsilon > 0$ such that $v'(r) > u'(r)$ for $r_m \leq r < r_m + \epsilon$ and hence $v(r) > u(r)$ for $r_m \leq r < r_m + \epsilon$. Let r_* be the first r greater than r_m for which $v'(r) = u'(r)$. Using $v(r_*) \geq u(r_*)$, $|w| \leq 1$, (9.4), (9.5), and (9.10), we see that $(u-v)''(r_*) > 0$ which contradicts $v'(r_*) = u'(r_*)$. Therefore $v'(r) > u'(r)$ for all $r \geq r_m$ which implies that

$$1 - w(r)^2 < v(r) \quad \forall\, r \geq r_m\,. \tag{9.11}$$

The general solution to (9.8) is

$$v = \frac{C_1}{r} + C_2 r^2 - \frac{1}{r}\int_0^r \rho^{1/2} w'(\rho)d\rho\,. \tag{9.12}$$

where C_1 and C_2 are arbitrary constants. So then

$$C_2 = \frac{1}{3r}\left(\frac{v}{r} + v' + \frac{w'}{r^{1/2}}\right)$$

and hence

$$\begin{aligned} C_2 &= \frac{1}{3r_m}\left(\frac{u(r_m)}{r_m} + u'(r_m) + \frac{w'(r_m)}{r_m^{1/2}}\right) && \text{by (9.9)} \\ &= 0 && \text{by (9.7)}\,. \end{aligned}$$

Therefore

$$1 - w^2(r) \leq \frac{C_1}{r} - \frac{1}{r}\int_0^r \rho^{1/2} w'(\rho)d\rho \quad \forall\, r \geq r_m\,.$$

As $w' \in \mathrm{o}(r^{-1})$ it is easy to see that there exists a $C > 0$ such that

$$1 - w^2(r) \leq \frac{C}{r^{1/2}} \quad \forall\, r \geq r_m\,.$$

Using the same arguments as in lemma 9.4 it follows from this inequality that $w' \in \mathrm{O}(r^{-3/2})$. □

Since $\lim_{r\to\infty} w(r) = 1$, we have

$$1 - w(r) = \int_r^\infty w'(\rho)d\rho$$

and hence

$$|1 - w(r)| \leq \int_r^\infty |w'(\rho)| d\rho \,. \tag{9.13}$$

But $w' \in \mathrm{O}(r^{-3/2})$ by lemma 9.6, and hence $1 - w(r) \in \mathrm{O}(r^{-1/2})$ by (9.13). Writing (7.15) as

$$w'' = -\alpha' w' + \frac{(w-1)(w+1)w}{r^2} \tag{9.14}$$

we see that $w'' \in \mathrm{O}(r^{-5/2})$ since $|w| \leq 1$, $w' \in \mathrm{O}(r^{-3/2})$, $1 - w(r) \in \mathrm{O}(r^{-1/2})$, and $\alpha' \in \mathrm{O}(r^{-2})$. Using (7.16), $\lim_{r\to\infty} \alpha(r) = \infty$, and similar arguments, it is straightforward to show that $\alpha \in \mathrm{O}(r^{-1})$ and $\alpha'' \in \mathrm{O}(r^{-2})$. □

Theorem 9.7. *There exists a sequence* $n = 1, 2, 3, \ldots$ *of* C^2 *solutions* $(\lambda_n, \mathfrak{U}_n, A^n, \psi^n)$ *to the EYMd equations* (2.16)*,* (2.17)*,* (2.22)*, and* (2.23)*.*

Proof. Let $(w_n(r), \alpha_n(r))$ be as in theorem 9.1. Define

$$W_i^n(\mathbf{x}) := \frac{w_n(|\mathbf{x}|) - 1}{r^2} \epsilon_i{}^j{}_k x^k \tau_j \ .$$

Then it follows from theorem 9.1, remark 9.2, and proposition 9.3 that for $n = 1, 2, 3, \ldots$ (W^n, α_n) is a C^2-solution to the flat space YMd equations, $w_n(r)$ satisfies the the hypotheses of proposition 5.7 for any $\sigma \in (0, 1/2]$, and $\alpha_n \in \mathcal{D}^{2,p}_{\delta_1}$ for any $\delta_1 \in (-1, 0)$ and $p > 3$. The proof then follows from proposition 8.2. □

Acknowledgements. I would like to thank R. Bartnik for helpful discussions and advice. This work was partially supported by the ARC grant A00105048 at the University of Canberra.

References

[1] R. Bartnik, *The Mass of an Asymptotically Flat Manifold*, Comm. Pure Appl. Math. **39** (1986), 661–693.

[2] R. Bartnik, *The spherically symmetric Einstein Yang-Mills equations*, Relativity Today (Z. Perjés, ed.), 1989, Tihany, Nova Science Pub., Commack NY, 1992, pp. 221–240.

[3] P. Bizon, *Saddle-point solutions in Yang-Mills-dilaton theory*, Phys. Rev. D **47** (1993) 1656.

[4] P. Bizon, *Saddle points of stringy action*, Acta Physica Polonica B **24** (1993), 1209.

[5] Y. Choquet-Bruhat and D. Christodoulou, *Elliptic systems in $H_{s,\delta}$ spaces on manifolds which are euclidean at infinity*, Acta. Math. **146** (1981), 129–150.

[6] S. Coleman, *Classical lumps and their quantum descendants*, New Phenomena in Subnuclear Physics, ed A Zichichi, Plenum, New York, 1977.

[7] S. Deser, *Absence of static solutions in source-free Yang-Mills theory*, Phys. Lett. **64B** (1976), 463-4.

[8] M. Giaquinta, *Introduction to Regularity Theory for Nonlinear Elliptic Systems*, Birkhäuser, Berlin, 1993.

[9] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second order*, 2$^{\text{nd}}$ ed., Revised 3$^{\text{rd}}$ printing, Springer, Berlin, 1998.

[10] S.P. Hastings, J.B. McLeod, and W.C. Troy, *Static spherically symmetric solutions of a Yang-Mills field coupled to a dilaton*, Proc. Roy. Soc. London Ser. A **449** (1995), 479–491.

[11] U. Heilig, *Existence of Rotating Stars in General Relativity*, Comm. Math. Phys. **166** (1995), 457–493.

[12] G. Lavrelashvili and D. Maison, *Static spherically symmetric solutions of a Yang-Mills field coupled to a dilaton*, Phys. Lett. B **295** (1992), 67–72.

[13] G. Lavrelashvili and D. Maison, *Regular and black hole solutions of Einstein-Yang-Mills-dilaton theory*, Nucl. Phys. B **410** (1993), 407.

[14] M. Lottermoser, *A convergent Post-Newtonian Approximation for the Constraint equations in General Relativity*, Ann. Inst. Heri Poincar'e **57** (1992), 279–317.